Group representations
Exercise sheet 2

`https://www.maths.tcd.ie/~mascotn/teaching/2023/MAU34104/index.html`

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THIS ASSIGNMENT IS NOT MANDATORY.

Email your answers to mascotn@tcd.ie by Friday Feburary 24, 13:00.

Exercise 1 *Preservation of semi-simplicity (50 pts)*

In this exercise, all modules are over a fixed ring R, and all modules are Artinian[1], meaning that there cannot exist an infinite descending chain of submodules.

1. (20 pts) Prove that a submodule of a semi-simple module is also semi-simple.

2. (20 pts) Prove that if $f : M \longrightarrow N$ is a module morphism, and if M is semi-simple, then $\operatorname{Im} f$ is also semi-simple.

3. (10 pts) Let now G be a group, K a field, and $f : V \longrightarrow W$ be a morphism between representations of G over K of finite degree. Prove that if V is semi-simple, then so are $\operatorname{Ker} f$ and $\operatorname{Im} f$.

Solution 1

We will constantly use the result that a module is semi-simple iff. every submodule admits a supplement.

1. Let M be a semi-simple module, and let $N \subseteq M$ be a submodule. If $P \subset N$ is a submodule of N, then it is also a submodule of M; as M is semi-simple, there exists a submodule $S \subseteq M$ such that $M = P \oplus S$, so that every $m \in M$ can be uniqueley decomposed as $m = p + s$ with $p \in P$ and $s \in S$. Let $S' = S \cap N$; this is a submodule of N. If $n \in N$, then also $n \in M$, so we can write $n = p + s$ with unique $p \in P$ and $s \in S$. Then $s = n - p \in N$ as $P \subseteq N$, so actually $s \in S \cap N = S'$. This shows that $N = P \oplus S'$, whence the result.

2. Since M is semi-simple, the submodule $\operatorname{Ker} f \subseteq M$ admits a supplement M'.

   Restricting f to M' and corestricting to $\operatorname{Im} f$ yields $f' = f_{|M'} : M' \longrightarrow \operatorname{Im} f$, which is injective since if $m \in \operatorname{Ker} f'$, then $m \in \operatorname{Ker} f \cap M'$ whence $m = 0$, and surjective, as if $n \in \operatorname{Im} f$, then $n = f(m)$ for some $m \in M$, which can be (uniquely) decomposed as $m = k + m'$ with $k \in \operatorname{Ker} f$ and $m' \in M'$, but then $n = f(m) = f(k + m') = f(k) + f(m') = f(m')$ as $k \in \operatorname{Ker} f$.

   Therefore f' is an isomorphism between M' and N. As M' is semi-simple by the previous question, so is N.

[1] NB we make this assumption to make our lives easier, but it can be shown that the properties established in this exercise actually remain valid without this assumption.

3. We can view V and W as R-modules where $R = K[G]$, which are Artinian since a descending chain of submodules would in particular be a descending chain of K-subspaces, and thus cannot be infinite (consider dimensions); and we can view f as a module morphism. Then $\operatorname{Ker} f$ is a submodule of V, and is therefore semi-simple by the first question, whereas $\operatorname{Im} f$ is a submodule of W, which is semi-simple by the second question.

Exercise 2 *A non-semi-simple ring (50 pts)*

Let K be a field, and let G be a finite group of order $n = \#G$. We will sometimes see the group ring $K[G] = \{\sum_{g \in G} \lambda_g e_g \mid \lambda_g \in K \text{ for all } g \in G\}$ as a module over itself.

1. (5 pts) Let $\Sigma = \sum_{g \in G} e_g \in K[G]$. Prove that $e_h \Sigma = \Sigma$ for all $h \in G$.
2. (5 pts) Prove that $S = \{\lambda\Sigma, \ \lambda \in K\}$ is a sub-$K[G]$-module of $K[G]$.
3. (5 pts) Identify S as a representation of G.

From now on, we assume that $n = 0$ in K.

4. (5pts) Prove that $\Sigma^2 = 0$ in $K[G]$.
5. (10pts) Deduce that $1 - \lambda\Sigma$ is invertible in $K[G]$ for all $\lambda \in K$, where $1 = e_{1_G}$ is the multiplicative identity of $K[G]$.

   Note that since $K[G]$ is not commutative in general, you must prove that your inverse works on both sides.

   Hint: For $x \in \mathbb{R}$ and $m \in \mathbb{N}$, what is the formula for the geometric series $1 + x + x^2 + \cdots + x^m$? How do you prove it?
6. (20 pts) Deduce that $K[G]$, viewed as a $K[G]$-module, is not semi-simple.

Solution 2

1. For all $h \in G$, we have

$$e_h \Sigma = e_h \sum_{g \in G} e_g = \sum_{g \in G} e_{hg} = \sum_{g \in G} e_g = \Sigma$$

   as $\begin{array}{ccc} G & \longrightarrow & G \\ g & \longmapsto & hg \end{array}$ is a bijection.

2. For all $\sum_{g \in G} \lambda_g e_g \in K[G]$ we have

$$\left(\sum_{g \in G} \lambda_g e_g \right) \Sigma = \sum_{g \in G} \lambda_g (e_g \Sigma) = \sum_{g \in G} \lambda_g \Sigma = \left(\sum_{g \in G} \lambda_g \right) \Sigma,$$

   which proves that $M = \{\lambda\Sigma, \ \lambda \in K\} \subset K[G]$ is closed by multiplication by all "scalars" in $K[G]$. Since it is also an additive subgroup of $K[G]$ (and even a K-subspace), it is a submodule of $K[G]$.

3. We can therefore view M as a representation of G over K. Its degree is $\dim_K M = 1$, and since for all $h \in G$, $e_h\Sigma = \Sigma$, we see that G acts trivially on it. Thus this is the trivial representation $\mathbb{1}$.

4. We compute that
$$\Sigma^2 = \left(\sum_{g\in G} e_g\right)\Sigma = \sum_{g\in G} e_g\Sigma = \sum_{g\in G}\Sigma = n\Sigma$$
as we have proved that $e_g\Sigma = \Sigma$ for all $g \in G$ in the previous question. As $n = 0$ in K and therefore in $K[G]$, the result follows.

5. Expanding shows that $(1-x)(1+x+x^2+\cdots+x^n) = 1-x+x-x^2+\cdots-x^n+x^n-x^{n+1} = 1-x^{n+1}$ for all $x \in \mathbb{R}$, and in fact this identity remains valid in any ring, even if it is not commutative, since powers of x and 1 always commute with each other. In particular, for $n = 1$ we have that $(1-x)(1+x) = 1-x^2$, so $(1-\lambda\Sigma)(1+\lambda\Sigma) = 1-\lambda^2\Sigma^2 = 1$ as $\Sigma^2 = 0$, and similarly $(1+\lambda\Sigma)(1-\lambda\Sigma) = 1-\lambda^2\Sigma^2 = 1$. This shows that $1-\Sigma$ is invertible in $K[G]$, with inverse $1+\Sigma \in K[G]$.

6. Suppose by contradiction that $K[G]$ is semi-simple. An infinite descending chain of submodules of $K[G]$ would in particular be a descending chain of K-subspaces, which cannot exist (consider dimensions). Therefore, the submodule M of $K[G]$ would admit a supplement S, that is to say $K[G] = M \oplus S$. Then we could decompose (uniquely) $1 = m + s$ with $m \in M$ and $s \in S$, so we would have $m = \lambda\Sigma$ for some $\lambda \in K$ and $s = 1 - m = 1 - \lambda\Sigma$. As shown in the previous question, there exists $t \in K[G]$ such that $ts = 1$; but then $\Sigma = \Sigma 1 = (\Sigma t)s$ would lie in S since $\Sigma t \in K[G]$, $s \in S$, and S is a submodule; therefore $0 \neq \Sigma \in M \cap S$, which contradicts $K[G] = M \oplus S$ as we would have the two decompositions $\Sigma = \Sigma + 0 = 0 + \Sigma$.

These were the only mandatory exercises, that you must submit before the deadline. The following exercise is not mandatory; it is not worth any points, and you do not have to submit them. However, you can try to solve it for practice, and you are welcome to email me if you have questions about them. The solution will be made available with the solution to the mandatory exercises.

Exercise 3 *Annihilators and simple modules*

Let R be a ring, which need not be commutative. We say that $I \subseteq R$ is a *left ideal* if it is an additive subgroup of $(R, +)$ and if $ri \in I$ for all $r \in R$ and $i \in I$. We define *right ideals* similarly. If I is both a left ideal and a right ideal, then we say that it is a *two-sided* ideal. A *maximal* left ideal is a left ideal $M \neq R$ such that there are no left ideals I such that $M \subsetneq I \subsetneq R$.

1. Let M be an R-module. Define its *annihilator* as

$$\operatorname{Ann} M = \{r \in R \mid rm = 0 \text{ for all } m \in M\} \subseteq R.$$

Prove that $\operatorname{Ann} M$ is a *two-sided* ideal of R.

2. The ring R can be viewed as a module over itself; we denote this module by ${}_R R$, so as to clearly distinguish between the ring R and the R-module ${}_R R$. Identify the submodules of ${}_R R$.

3. Let S be a module, and let $s \in S$.

   (a) Prove that the map
   $$\begin{array}{rcl} f_s : {}_R R & \longrightarrow & S \\ r & \longmapsto & rs \end{array}$$
   is a module morphism.

   (b) Prove that S is simple iff. f_s is surjective for all $s \neq 0$.

   (c) Deduce that if S is simple, then it is isomorphic to ${}_R R/M$, where M is a maximal left ideal of R.

   (d) Prove that conversely, if M is a maximal left ideal of R, then ${}_R R/M$ is a simple R-module.

4. Beware that in general, the annihilator of ${}_R R/M$, which is a two-sided ideal, does not agree with the left ideal M! Here is an example: Take $R = M_2(\mathbb{R})$, and $S = \mathbb{R}^2$, which is an R-module if we view its elements as column vectors. Prove that S is simple, determine $\operatorname{Ann} S$, and find a maximal left ideal M of R such that $S \simeq {}_R R/M$.

Solution 3

1. Let $m \in M$. Then $0m = 0$, so $0 \in \operatorname{Ann} M$. Besides, if $r, s \in \operatorname{Ann} M$, then $(r - s)m = rm - sm = 0 - 0 = 0$, so $r - s \in \operatorname{Ann} M$ as this holds for any $m \in M$. Finally, let $r \in \operatorname{Ann} M$ and $x \in R$. Then $(xr)m = x(rm) = x0 = 0$, and $(rx)m = r(xm) = 0$ as $xm \in M$ as M is a module; since these hold for any $m \in M$, $\operatorname{Ann} M$ is a two-sided ideal of R.

2. Let $M \subseteq {}_R R$ be a submodule. Then $M \subseteq (R, +)$ is an additive subgroup, and besides $xm \in M$ for all $x \in R$ and $m \in M$, so M is a left ideal. Conversely, we see that any left ideal of R is actually a submodule of ${}_R R$. So the submodules of ${}_R R$ are precisely the left ideals of R.

3. (a) f_s is additive since for all $r, r' \in {}_R R = R$, $f(r+r') = (r+r')s = rs+r's = f(r) + f(r')$, and linear because for all $\lambda \in R = {}_R R$ and $r \in {}_R R = R$, $f(\lambda r) = (\lambda r)s = \lambda(rs) = \lambda f(rs)$.

   (b) Note that $s = f_s(1_R) \in \operatorname{Im} f_s$. So if $s \neq 0$, then $\operatorname{Im} f$ is a nonzero submodule of S. So if S is simple, then this submodule must be all of S, so f_s is surjective.

   Conversely, suppose that S is not simple. Then it has a non-trivial submodule $\{0\} \subsetneq T \subsetneq S$. As $T \neq \{0\}$, we can find $0 \neq t \in T$; then $t \in S$, and $\operatorname{Im} f_t = \{rt, r \in R\} \subseteq T$ is strictly smaller that S as it is contained in T, so f_t is not surjective.

(c) As S is simple, $S \neq \{0\}$, so let $0 \neq s \in S$. The previous question ensures that f_s is surjective, so the isomorphism theorem for modules yields $S \simeq_R R/M$ where $M = \operatorname{Ker} f_s$. In particular, M is a submodule of ${}_RR$, and therefore a left ideal by the first question.

If M were not maximal, we could find another ideal (=submodule) M' subst that $M \subsetneq M' \subsetneq {}_RR$, and then M'/M would be a non-trivial submodule of R/M, which is absurd as $R/M \simeq S$ is simple, so M must be maximal.

(d) The submodules of the quotient ${}_RR/M$ are precisely the M'/M, where M' is a submodule (= left ideal) of ${}_RR$ which contains M. As M is maximal, there are exactly two such submodules, namlely M and ${}_RR$, so correspondingly ${}_RR/M$ has only two submodules, which are $M/M = \{0\}$ and ${}_RR/M$ =itself. This shows that ${}_RR/M$ is a simple module.

4. Let $e_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \in \mathbb{R}^2$, and $e_2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \in \mathbb{R}^2$; recall that for all $A \in M_2(\mathbb{R})$, Ae_1 is the left column of A, and Ae_2 is the right column of A.

Let now $0 \neq v \in \mathbb{R}^2$, and let $w \in \mathbb{R}^2$. There exists a matrix $B \in GL_2(\mathbb{R})$ whose first column is v, so that B^{-1} takes v to e_1, and a matrix $C \in M_2(\mathbb{R})$ whose first column is w, so that it takes e_1 to w; then $A = CB^{-1} \in M_2(\mathbb{R})$ satisfies $Av = w$. This shows that the map $f_v : A \mapsto Av$ is surjective for all $v \neq 0$, so by question 2, $\mathbb{R}^2$ is a simple $M_2(\mathbb{R})$-module. Still by question 2, for any $0 \neq v \in \mathbb{R}^2$, the $M_2(\mathbb{R})$-module $\mathbb{R}^2$ is isomorphic to $M_2(\mathbb{R})/N$, where N is the left ideal $\{A \in M_2(\mathbb{R}) \mid Av = 0\}$. For instance, if we take $v = e_1$, then

$$N = \{A \in M_2(\mathbb{R}) \mid Ae_1 = 0\} = \left\{ \begin{pmatrix} 0 & x \\ 0 & y \end{pmatrix} \;\middle|\; x, y \in \mathbb{R} \right\} = \begin{pmatrix} 0 & * \\ 0 & * \end{pmatrix}.$$

However, $\operatorname{Ann} \mathbb{R}^2$ is the set of matrices $A \in M_2(\mathbb{R})$ such that $Av = 0$ for all $v \in \mathbb{R}^2$; taking v to be e_1, we get that the left column Ae_1 of A must be 0, and then taking $v = e_2$, we see that the right column of A must also be 0. Thus

$$\operatorname{Ann} \mathbb{R}^2 = \{0\} \neq N.$$