Group representations
Exercise sheet 1

`https://www.maths.tcd.ie/~mascotn/teaching/2023/MAU34104/index.html`

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Email your answers to mascotn@tcd.ie by Friday Feburary 10, 13:00.

Exercise 1 *Decomposition over $\mathbb{R}$ (100 pts)*

You are allowed to use without proof the results from Exercises 2-4 to solve this exercise (but I highly recommend you try to solve Exercises 2-4 as well!).

In this exercise, we consider over $K = \mathbb{R}$ two representations of $G = S_3$ constructed in the lectures, namely the permutation representation

$$\mathrm{Perm} : S_3 \longrightarrow \mathrm{GL}_3(K)$$

induced by $S_3 \circlearrowright \{1, 2, 3\}$, and

$$\triangleleft : S_3 \longrightarrow \mathrm{GL}_2(K), \quad (123) \mapsto \begin{pmatrix} -1 & -1 \\ 1 & 0 \end{pmatrix}, \quad (12) \mapsto \begin{pmatrix} 1 & 1 \\ 0 & -1 \end{pmatrix}$$

obtained by labelling the vertices of an equilateral triangle by $\{1, 2, 3\}$.

1. (35 pts) Prove that $\triangleleft$ is irreducible. Is $\triangleleft$ indecomposable?
2. (65 pts) Prove that $\mathrm{Perm} \simeq \mathbb{1} \oplus \triangleleft$ as representations of S_3.

Solution 1

1. Since $\deg \triangleleft = 2$, if it were reducible, then it would have a subrepresentation of degree 1.

   We know that $(123) \in S_3$ acts on the representation space $\mathbb{R}^2$ of $\triangleleft$ by a rotation of angle $2\pi/3$; therefore, it does not stabilise any vector line, so $\triangleleft$ is irreducible.

   Here is a more algebraic[1] way to say this: by the previous exercise, a subrepresentation of $\triangleleft$ of degree 1 would be spanned by a common eigenvector for all $g \in S_3$, yet the matrix $\begin{pmatrix} -1 & -1 \\ 1 & 0 \end{pmatrix}$ of (123) has characteristic polynomial $x^2 + x + 1$, which has negative discriminant, so (123) has no eigenvector over $\mathbb{R}$.

[1] I do not mean to imply that this more algebraic way is better. In fact, I personally find the geometric argument more convincing!

Here is another proof, which shows that $\triangleleft$ is actually irreducible even over $\mathbb{C}$: the matrix $\begin{pmatrix}1 & 1\\ 0 & -1\end{pmatrix}$ clearly has eigenvalues 1 and -1, with respective eigenvectors $v_+ = \begin{pmatrix}1\\0\end{pmatrix}$ and $v_- = \begin{pmatrix}1\\-2\end{pmatrix}$; but since since neither is an eigenvector of (123), there is no common eigenvector for all S_3.

Anyway, we conclude that $\triangleleft$ is irreducible, and therefore also indecomposable.

2.
- Analysis: Let us define $\sigma = (123)$, $\tau = (12)$. The representation space of $\mathbb{1}$ is $\mathbb{R}$, with trivial action of S_3; let u be a vector which spans it. The representation space of $\triangleleft$ is $\mathbb{R}^2$; let (e_1, e_2) be its standard basis, on which the matrix of σ is $S = \begin{pmatrix}-1 & -1\\ 1 & 0\end{pmatrix}$ and that of τ is $T = \begin{pmatrix}1 & 1\\ 0 & -1\end{pmatrix}$. Finally, the representation space of Perm is $\mathbb{R}^3$, on which S_3 acts by permuting the coordinates.

  Suppose we have a representation morphism $f : \mathbb{1} \oplus \triangleleft \longrightarrow \text{Perm}$, and define $v = f(u)$, $f_1 = f(e_1)$, $f_2 = f(e_2) \in \mathbb{R}^3$.

  Since $gu = u$ for all $g \in S_3$, we must also have $gv = v$ for all $g \in S_3$; therefore v is a multiple of $\begin{pmatrix}1\\1\\1\end{pmatrix}$.

  We also observe that $\tau e_1 = e_1$, so $\tau f_1 = f_1$, which means that $f_1 = \begin{pmatrix}a\\a\\b\end{pmatrix}$ for some $a, b \in \mathbb{R}$. Furthermore, we have $\sigma e_2 = -e_1$, so $e_2 = -\sigma^{-1} e_1$, whence $f_2 = -\sigma^{-1} f_1 = \begin{pmatrix}-a\\-b\\-a\end{pmatrix}$. Besides, from the fact that T is triangular, we see that it has the eigenvalue -1, and we compute that the corresponding eigenspace is the span of $e_1 - 2e_2$, so that $\tau(e_1 - 2e_2) = -(e_1 - 2e_2)$ whence $\begin{pmatrix}a+2b\\3a\\b+2a\end{pmatrix} = \tau(f_1 - 2f_2) = -(f_1 - 2f_2) = \begin{pmatrix}-3a\\-a-2b\\-b-2a\end{pmatrix}$; this forces $b = -2a$.

- Synthesis: Let $f : \mathbb{1} \oplus \triangleleft \longrightarrow \text{Perm}$ be the linear transformation defined by

$$f(u) = v = \begin{pmatrix}1\\1\\1\end{pmatrix}, \quad f(e_1) = f_1 = \begin{pmatrix}1\\1\\-2\end{pmatrix}, \quad f(e_2) = f_2 = \begin{pmatrix}-1\\2\\-1\end{pmatrix}.$$

  Since $\begin{vmatrix}1 & 1 & -1\\ 1 & 1 & 2\\ 1 & -2 & -1\end{vmatrix} = 9 \neq 0 \in \mathbb{R}$, we see that this linear transformation is invertible. It remains to check if it is a representation morphism, that is to say if $f(gw) = gf(w)$ for all $w \in \mathbb{1} \oplus \triangleleft$ and $g \in S_3$. Clearly, it is sufficient to check this for w ranging over the basis (u, e_1, e_2) of $\mathbb{1} \oplus \triangleleft$, and also for $g = \sigma$ and τ since σ and τ generate S_3.

  By construction, we have $f(gu) = f(u) = v = gf(v)$ for all $g \in S_3$.

We also have $f(\tau e_1) = f(e_1) = f_1 = \tau f_1$ by construction, and we check that $f(\sigma e_1) = f(-e_1 + e_2) = -f_1 + f_2 = \begin{pmatrix} -2 \\ 1 \\ 1 \end{pmatrix} = \sigma f_1$.

And finally, we check that $f(\tau e_2) = f(e_1 - e_2) = f_1 - f_2 = \begin{pmatrix} 2 \\ -1 \\ -1 \end{pmatrix} = \tau f_2$, and we have $f(\sigma e_2) = f(-e_1) = -f_1 = \sigma f_2$ by construction.

In conclusion, the answer is **yes**, and we have exhibited an isomorphism of representations $f : \mathbb{1} \oplus \triangleleft \simeq \mathrm{Perm}$.

Remark: we can summarise the verification calculations that we have just performed by saying that the respective matrices of σ and τ on the basis (v, f_1, f_2) of $\mathbb{R}^3$ are

$$\left(\begin{array}{c|cc} 1 & 0 & 0 \\ \hline 0 & -1 & -1 \\ 0 & 1 & 0 \end{array}\right) = \begin{pmatrix} 1 & 0 \\ 0 & S \end{pmatrix} \quad \textit{and} \quad \left(\begin{array}{c|cc} 1 & 0 & 0 \\ \hline 0 & 1 & 1 \\ 0 & 0 & -1 \end{array}\right) = \begin{pmatrix} 1 & 0 \\ 0 & T \end{pmatrix}.$$

This way, we really "see" the representation isomorphism at work.

You may also have noticed that we had no special reason to take $a = 1$, nor to decide that v is not another multiple of $(1, 1, 1)$. Indeed, any other choice also "works". This demonstrates the existence of automorphisms of the representation $\mathbb{1} \oplus \triangleleft \simeq \mathrm{Perm}$ (and it can be shown that this is actually all of its automorphisms).

Finally, we note that it is possible to "see" the decomposition $\mathrm{Perm} \simeq \mathbb{1} \oplus \triangleleft$ purely geometrically: imagine an orthonormal frame in $\mathbb{R}^3$ with 3 mutually orthogonal unit vectors shooting from the origin, and a tilted plane resting on the tip of these 3 vectors. Drawing line segments joining these 3 vector tips, we get an equilateral triangle in this plane; and the line spanned by the vector $(1, 1, 1)$ goes through the origin and the centre of this triangle, where it shoots perpendicularly through this plane. As S_3 permutes these 3 vectors (by definition of Perm), the vertices of the triangle are also permuted, but the plane remains globally invariant, whence the subrepresentation $\triangleleft$, whereas the points of the line spanned by $(1, 1, 1)$ remain all fixed, whence the subrepresentation $\mathbb{1}$.

This is the only mandatory exercise, that you must submit before the deadline. The following exercises are not mandatory; they are not worth any points, and you do not have to submit them. However, I *strongly* recommend you try to solve them for practice, and you are welcome to email me if you have questions about them. The solutions will be made available with the solution to the mandatory exercise.

Exercise 2 *Representation morphisms form a subspace*

Let G be a group, K a field, and let $\rho_1 : G \longrightarrow \mathrm{GL}(V_1)$ and $\rho_2 : G \longrightarrow \mathrm{GL}(V_2)$ be two representations of G over K.

Recall that the set $\mathrm{Hom}(V_1, V_2)$ of all linear transformations from V_1 to V_2 has a vector space structure, with addition and scalar multiplications defined pointwise (that is to say if $T, U \in \mathrm{Hom}_(V_1, V_2)$ and $\lambda \in K$, then $T + U$ is defined as $(T + U)(v_1) = T(v_1) + U(v_1)$ for all $v_1 \in V_1$, and λT is defined as $(\lambda T)(v_1) = \lambda\big(T(v_1)\big)$ for all $v_1 \in V_1$).

Prove that the subset $\mathrm{Hom}_G(V_1, V_2)$ consisting of linear transformations which are representation morphisms is a subspace of $\mathrm{Hom}(V_1, V_2)$.

Solution 2

We must prove that whenever $T, U \in \mathrm{Hom}_G(V_1, V_2)$ and $\lambda \in K$, $T + U$ and λT also lie in $\mathrm{Hom}_G(V_1, V_2)$.

First of all, recall that $\mathrm{Hom}_G(V_1, V_2)$ is defined as the set of $T \in \mathrm{Hom}(V_1, V_2)$ such that for all $g \in G$ and $v_1 \in V_1$,

$$T\big(\rho_1(g)(v_1)\big) = \rho_2(g)\big(T(v_1)\big).$$

So let $T, U \in \mathrm{Hom}_G(V_1, V_2)$, and let $v_1 \in V_1$ and $g \in G$. Then

$$\begin{aligned}
(T + U)\big(\rho_1(g)(v_1)\big) &= T\big(\rho_1(g)(v_1)\big) + U\big(\rho_1(g)(v_1)\big) \text{ by definition of } T + U \\
&= \rho_2(g)\big(T(v_1)\big) + \rho_2(g)\big(U(v_1)\big) \text{ as } T, U \in \mathrm{Hom}_G(V_1, V_2) \\
&= \rho_2(g)\big(T(v_1) + U(v_1)\big) \text{ as } \rho_2(g) : V_2 \to V_2 \text{ is linear for all } g \in G \\
&= \rho_2(g)\big((T + U)(v_1)\big) \text{ by definition of } T + U,
\end{aligned}$$

which proves that $T + U \in \mathrm{Hom}_G(V_1, V_2)$. Similarly, if $\lambda \in K$, $T \in \mathrm{Hom}_G(V_1, V_2)$, $v_1 \in V_1$, and $g \in G$, then

$$\begin{aligned}
(\lambda T)\big(\rho_1(g)(v_1)\big) &= \lambda\Big(T\big(\rho_1(g)(v_1)\big)\Big) \text{ by definition of } \lambda T \\
&= \lambda\Big(\rho_2(g)\big(T(v_1)\big)\Big) \text{ as } T \in \mathrm{Hom}_G(V_1, V_2) \\
&= \rho_2(g)\Big(\lambda\big(T(v_1)\big)\Big) \text{ as } \rho_2(g) : V_2 \to V_2 \text{ is linear for all } g \in G \\
&= \rho_2(g)\big((\lambda T)(v_1)\big) \text{ by definition of } \lambda T.
\end{aligned}$$

Exercise 3 *Representations of degree 1*

Prove that a representation of degree 1 is always indecomposable and irreducible.

Solution 3

Let $\rho : G \longrightarrow \mathrm{GL}(V)$ be a representation of degree 1. This means that $\dim V = 1$, so V has no subspaces apart from $\{0\}$ and itself. In particular, ρ does not have any subrepresentatinos apart from $\{0\}$ and V, which proves that ρ is irreducible.

Since irreducibility always implies indecomposability, it follows that ρ is also indecomposable.

Exercise 4 *Subrepresentations of degree 1*

Let G be a group, K be a field, and V a representation of G over K. Prove that there exists a subrepresentation of V of degree 1 iff. there exists a nonzero vector $v \in V$ which is an eigenvector for all $g \in G$ (possibly with an eigenvalue which depends on g).

Solution 4

Let $V' \subset V$ be a subrepresentation of degree 1. Then V' is spanned by a nonzero vector v, so $V' = \{\lambda v, \ \lambda \in K\}$. Since V' is a subrepresentation, for each $g \in G$ we have $gv \in V'$, so $gv = \lambda_g v$ for some $\lambda_g \in K$, which means that v is an eigenvector for all $g \in G$.

Conversely, suppose $0 \neq v \in V$ is an eigenvector for all $g \in G$, so that for all $g \in G$ we have $gv = \lambda_g v$ for some $\lambda_g \in K$. Then for all $g, h \in G$, $ghv = g\lambda_h v = \lambda_h gv = \lambda_h \lambda_g v$ so $\lambda_{gh} = \lambda_g \lambda_h$; besides $\lambda_{1_G} = 1$ so, taking $h = g^{-1}$, we must have $\lambda_g \neq 0$ for all $g \in G$. This shows that $g \mapsto \lambda_g$ is a group morphism $G \longrightarrow K^\times = \mathrm{GL}_1(K)$, so that the span of v in V is a subrepresentation.

The following exercise has been included for those of you who wish to try their skills in more "exotic" situations. It is not meant to be as profitable for your understanding of the material of this module as the previous exercises. You are still welcome to try to solve it for practice, and to email me if you have questions about it. The solution will be also made available with the solution to the other exercises.

Exercise 5 *Decomposition over $\mathbb{Z}/p\mathbb{Z}$*

Redo Exercise 1, but with $K = \mathbb{Z}/p\mathbb{Z}$ instead of $\mathbb{R}$, with $p \in \mathbb{N}$ prime. Is $\triangleleft$ still irreducible? Indecomposable? Do we still have $\mathrm{Perm} \simeq \mathbb{1} \oplus \triangleleft$?
Your answers may depend on the value of p; explore all cases!

Solution 5

1. We can still argue that if $\triangleleft$ is reducible, then it has a subrepresentation of degree 1, and thus a common eigenvector for $S = \begin{pmatrix} -1 & -1 \\ 1 & 0 \end{pmatrix}$ and $T = \begin{pmatrix} 1 & 1 \\ 0 & -1 \end{pmatrix}$. The eigenvalues of the triangular matrix T are ± 1. If $p \neq 2$, they are disctinct, so T is diagonalisable, with eigenspaces spanned respectively by $v_+ = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ and $v_- = \begin{pmatrix} 1 \\ -2 \end{pmatrix}$; thus the subrepresentation must be one of these. We compute that $Sv_+ = \begin{pmatrix} -1 \\ 1 \end{pmatrix}$ is not collinear to $v_=$ for any p because of its second component, so it never spans a subrepresentation; and that $Sv_- = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$. If

Sv_- were collinear to v_-, then by the first coefficient we would actually have $Sv_- = v_-$, which happens iff. $p = 3$. So for $p \geq 5$, $\triangleleft$ is still irreducible and in particular indecomposable, whereas for $p = 3$, $\triangleleft$ is reducible, but still indecomposable as we have proved that it has only one subrepresentation of degree 1 (we would need two of them to decompose $\triangleleft$).

Finally, if $p = 2$, then the only eigenvalue of T is $1 = -1$, so T is not diagonalisable (else T would be conjugate to the identity, and therefore equal to the identity). its egeinspace is still spanned by v_+, and as Sv_+ is not collinear to v_+, we conclude that $\triangleleft$ is still irreducible (and in particular, indecomposable) for $p = 2$.

2. No matter what the value of p is, the computations done in the Synthesis part in the answer to the previous exercise show that the map f defined there is still a morphism of representations from $\mathbb{1} \oplus \triangleleft$ to Perm. Besides, it is an isomorphism provided that $\begin{vmatrix} 1 & 1 & -1 \\ 1 & 1 & 2 \\ 1 & -2 & -1 \end{vmatrix} = 9 \neq 0$, i.e. that $p \neq 3$. So the answer is still yes if $p \neq 3$.

   Let us now suppose that $p = 3$, and let us take a look again at $v_- = e_1 - 2e_2 = e_1 + e_2$. We still have $Tv_- = -v_-$, but now that $p = 3$, we also have $Sv_- = v_-$. Therefore, if Perm were isomorphic to $\mathbb{1} \oplus \triangleleft$, it would contain a nonzero vector which is negated by (12) and thus of the form $\begin{pmatrix} c \\ -c \\ 0 \end{pmatrix}$ with $0 \neq c \in \mathbb{Z}/3\mathbb{Z}$, and fixed by (123) which is a contradiction. So for $p = 3$, the answer is no!

Actually, we can prove that Perm is indecomposable when $p = 3$. This immediately implies that Perm $\not\simeq \mathbb{1} \oplus \triangleleft$. Although of course this was not required by the exercise, this has the advantage of answering the question in a less "magic" way than the previous approach, which throws in v_- without really explaining where the idea comes from.

First of all, observe that

$$x^3 = x \text{ for all } x \in \mathbb{Z}/3\mathbb{Z} = \{0, 1, 2 = -1\}. \tag{1}$$

If Perm decomposed, it would be either into a direct sum of two representations of degrees 1 and 2, or three of degree 1. Putting two of them together in the latter case, we may assume that Perm decomposes as $L \oplus P$ with $\deg L = 1$, $\deg P = 2$.

Let $S = \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}$ be the matrix of (123) on the standard basis of Perm. Its characteristic polynomial is $x^3 - 1$, so its only eigenvalue is 1 by (1), and we find that the corresponding eigenspace is spanned by $\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$. As L must be spanned by an eigenvector of S, it must be this eigenspace.

Now that L is determined, let us focus on P. It is a hyperplane, so of the form $P = \operatorname{Ker} l$ for some linear form $l \neq 0$ (i.e. it is defined by a single linear equation, which is not the trivial equation $0 = 0$), which is unique up to scaling. Let $\begin{pmatrix} a & b & c \end{pmatrix}$ be the matrix of l (i.e. $l(x, y, z) = ax+by+cz$). Since P is stable by S_3 and in particular by (123), we must have $\operatorname{Ker} l = P = \operatorname{Ker}(lS)$, so $lS = \begin{pmatrix} a & b & c \end{pmatrix}$ is proportional to l. This means that there exists $0 \neq \lambda \in \mathbb{Z}/3\mathbb{Z}$ such that $lS = \lambda l$, whence $b = \lambda a$, $c = \lambda b$, $a = \lambda c$. This implies $a = \lambda^3 a$, $b = \lambda^3 b$, $c = \lambda^3 c$; since a, b, c are not all 0, we must have $\lambda^3 = 1$, whence $\lambda = 1$ in view of (1). Thus $a = b = c$, so P must be the hyperplane defined by $x + y + z = 0$. But then L and P are not in direct sum since $L \subset P$, absurd.