Coláiste na Tríonóide, Baile Átha Cliath
Trinity College Dublin
Ollscoil Átha Cliath | The University of Dublin

Faculty of Engineering, Mathematics and Science
School of Mathematics

JS/SS Maths/TP/TJH **Semester 2, 2020–2021**

MAU34104 Group representations

Thursday 20 May 2021 **Online exam** **12:00PM – 6:00PM**

Dr. Nicolas Mascot
mascotn@tcd.ie

Instructions that apply to all take-home exams:

1. This is an open-book exam. You are allowed to use your class notes, textbooks and any material which is directly linked from the module's Blackboard page or from the module's webpage, if it has one. You may not use any other resources except where your examiner has specifically indicated in the "Additional instructions" section below. Similarly, you may only use software if its use is specifically permitted in that section. You are not allowed to collaborate, seek help from others, or provide help to others.

2. If you have any questions about the content of this exam, you may seek clarification from the lecturer using the e-mail address provided. You are not allowed to discuss this exam with others. You are not allowed to send exam questions or parts of exam questions to anyone or post them anywhere.

3. Unless otherwise indicated by the lecturer in the "Additional instructions" section, solutions must be submitted through Blackboard in the appropriate section of the module webpage by the deadline listed above. You must submit a single pdf file for each exam separately and sign the following declaration in each case. It is your responsibility to check that your submission has uploaded correctly in the correct section.

Additional instructions for this particular exam:

All the representations considered in this exam are over the field $\mathbb{C}$ of the complexes.

This exam is made up of **three** Questions. Attempt **all three** Questions.

Typesetting your answers in LaTeX is allowed, but will not result in any bonus marks.

The use of non-programmable calculators is allowed.

Question 1 *Homotheties and representations (30 pts)*

Let G be a finite group, let $\rho : G \longrightarrow \mathrm{GL}(V)$ be a complex representation of G of degree $d \in \mathbb{N}$, and let χ be the character of ρ. A *homothety* of V is a linear transformation of V of the form $\lambda \operatorname{Id}_V$ for some $\lambda \in \mathbb{C}$; its matrix is therefore the diagonal matrix $\begin{pmatrix} \lambda & & 0 \\ & \ddots & \\ 0 & & \lambda \end{pmatrix}$ with respect to any basis of V.

1. (5 pts) Let $g \in G$. Prove that $\rho(g)$ is a homothety of V if and only if $|\chi(g)| = d$.

2. (3 pts) Let $Z(\chi) = \{g \in G \mid \rho(g) \text{ is a homothety of } V\}$. Prove that $Z(\chi)$ is a normal subgroup of G.

3. (10 pts) The *centre* of G is the normal subgroup $Z(G)$ of G defined as

   $$Z(G) = \{z \in G \mid zg = gz \text{ for all } g \in G\}.$$

   Prove that if ρ is irreducible, then $Z(G) \subseteq Z(\chi)$.

   Hint: Schur.

4. (12 pts) Prove that $Z(G) = \bigcap_{\psi \in \mathrm{Irr}(G)} Z(\psi)$.

   Hint: Let $z \in \bigcap_{\psi \in \mathrm{Irr}(G)} Z(\psi)$ *and* $g \in G$*, and consider the element* $h = zgz^{-1}g^{-1} \in G$*.*

Question 2 *The affine group* $\mathrm{AGL}(1,5)$ *(44 pts)*

Let $\mathbb{F}_5$ be the field $\mathbb{Z}/5\mathbb{Z} = \{0, 1, 2, 3 = -2, 4 = -1\}$, and let G be the set of polynomials $f_{a,b} = ax + b$ with $a, b \in \mathbb{F}_5$, $a \neq 0$. We equip G with the law given by composition:

$$(f_{a,b} \cdot f_{a',b'})(x) = f_{a,b}\big(f_{a',b'}(x)\big).$$

We admit that G is a group, whose identity is $f_{1,0}$ and whose conjugacy classes are:

- $\{f_{1,0}\}$,
- $\{f_{1,b} \mid b \neq 0\}$,
- $\{f_{2,b} \mid b \in \mathbb{F}_5\}$,
- $\{f_{-2,b} \mid b \in \mathbb{F}_5\}$,
- $\{f_{-1,b} \mid b \in \mathbb{F}_5\}$.

1. (1 pt) Prove that the map
$$\begin{aligned} \lambda : G &\longrightarrow \mathbb{F}_5^\times \\ f_{a,b} &\longmapsto a \end{aligned}$$
is a group morphism.

2. (8 pts) Deduce the existence of four pairwise non-isomorphic irreducible representations of G of degree 1, and write down their characters.

   Hint: $\mathbb{F}_5^\times$ *is cyclic and generated by* 2.

3. (5 pts) Determine the number of irreducible representations of G, and their degrees.

4. (12 pts) Determine the character table of G.

5. (8 pts) We have a natural action of G on $\mathbb{F}_5$ defined by $f_{a,b} \cdot x = f_{a,b}(x)$. Determine up to isomorphism the decomposition into irreducible representations of the permutation representation $\mathbb{C}[\mathbb{F}_5]$ attached to this action of G on $\mathbb{F}_5$.

Question continues on the next page

6. (10 pts) The element $f_{-1,0}$ of G has order 2, and thus generates a subgroup $H = \{f_{1,0}, f_{-1,0}\}$ of G isomorphic to $\mathbb{Z}/2\mathbb{Z}$. Let $\varepsilon : H \longrightarrow \mathrm{GL}_1(\mathbb{C})$ be the degree 1 representation of H such that $\varepsilon(f_{-1,0}) = -\mathrm{Id}_{\mathbb{C}}$. Determine up to isomorphism the decomposition into irreducible representations of the induced representation $\mathrm{Ind}_H^G(\varepsilon)$.

Question 3 *The determinant of the character table (26 pts)*

Let G be a finite group of order $n \in \mathbb{N}$. We arrange the conjugacy classes $C_1, \cdots, C_r$ and the irreducible characters $\chi_1, \cdots, \chi_r$ of G arbitrarily, and we view the character table of G as an $r \times r$ complex matrix X. In other words, for all $1 \leq i, j \leq r$, the i, j-coefficient $X_{i,j}$ of X is $\chi_i(g_j)$, where g_j is any element of C_j.

1. (a) (6 pts) Prove that the complex conjugate of a character is a character.

   (b) (6 pts) Prove that the complex conjugate of an **irreducible** character is an **irreducible** character.

   (c) (4 pts) Deduce that $\overline{\det X} = \pm \det X$. What does this imply about the complex number $\det X$?

2. (a) (8 pts) Let D be the diagonal matrix whose j, j-coefficient is $\#C_j$ for all $1 \leq j \leq r$. Express the orthogonality relations in terms of the matrices X and D.

   (b) (4 pts) Deduce the value of $|\det X|$ in terms of n and of the $\#C_j$.

Question 2 *The mysterious group (50 pts)*

While exploring a temple, Indiana Jones has stumbled upon the following markings on a wall:

	C_1	C_2	C_3	C_4	C_5	C_6
			$\vdots$			
α	3	-1	0	1	z	$\bar{z}$
β	3	-1	0	1	$\bar{z}$	z
γ	6	2	0	0	-1	-1
δ	7	-1	1	-1	0	0
ϵ	8	0	-1	0	1	1

This is the character table of some mysterious group G, where z and $\bar{z}$ are the complex-conjugate roots of $x^2 + x + 2$. Unfortunately, the top of the table is damaged, so that the information about the conjugacy classes of G is unreadable, and that some of the first few rows of the table may be missing; fortunately, there is no more damage, so Indiana can clearly see that there are exactly 6 conjugacy classes (in other words, no columns have been erased). In spite of these difficulties, Indiana has taken upon himself to call the conjugacy classes $C_1, \cdots, C_6$, and to name $\alpha, \cdots, \epsilon$ the irreducible characters that are still readable.

1. (5 pts) How many irreducible characters are missing? Write them down (if any).

   Suggestion: you may find it useful to write down the completed character table before you solve the next questions.

2. (5 pts) Prove that the conjugacy class of the identity element 1_G of G is C_1.

3. (8 pts) Prove that G has exactly 168 elements.

4. (8 pts) Prove that G is a simple group.

5. (8 pts) Determine the number of elements of each of the conjugacy classes $C_1, \cdots, C_6$. In particular, prove that $\#C_2 = 21$.

Question continues on the next page

6. An inscription next to the table claims that all the elements of the conjugacy class C_2 have order 2 (so we admit this without proof). Let $h \in C_2$ be such an element, so that

$$H = \{1_G, h\}$$

is a subgroup of G isomorphic to $\mathbb{Z}/2\mathbb{Z}$, and let $\psi : H \longrightarrow \mathbb{C}$ be the character of H defined by

$$\psi(1_G) = 1, \ \psi(h) = -1$$

(you are not required to prove that this is indeed a character of H).

(a) (8 pts) Write down the induced character $\mathrm{Ind}_H^G \psi$ of G.

(b) (8 pts) Determine the decomposition of $\mathrm{Ind}_H^G \psi$ into irreducible characters of G.