Group representations
Exercise sheet 5

https://www.maths.tcd.ie/~mascotn/teaching/2023/MAU34104/index.html

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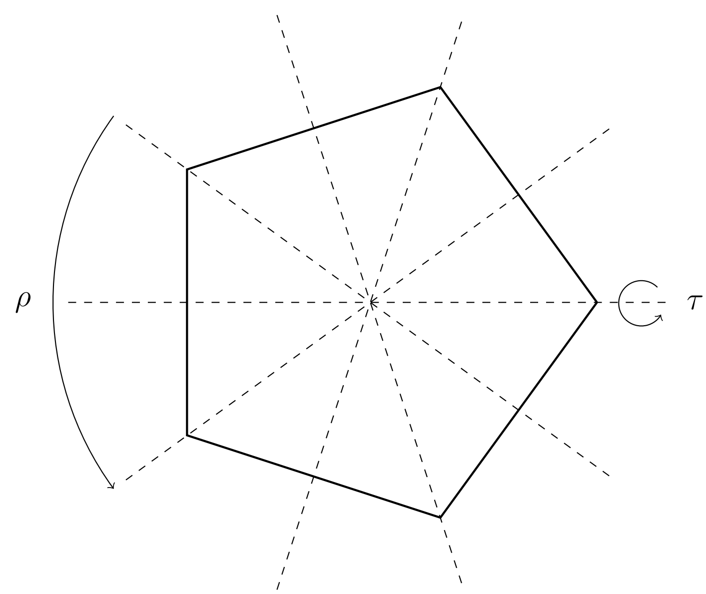
Email your answers to mascotn@tcd.ie by Tuesday April 11th, 12:00.

Exercise 1 *The character table of D_{2n}, n odd (100 pts)*

In this exercise, we fix an *odd* integer $n = 2m + 1 \geq 3$, and we consider the dihedral group $G = D_{2n}$ of transformations of the plane $\mathbb{R}^2$ that leave a regular n-gon invariant.

This group has $2n$ elements, n of which (including the identity) are rotations, whereas the other n are axial symmetries across lines joining a vertex to the midpoint of the opposite edge. We call ρ the rotation of angle $2\pi/n$, which has this order n, and τ one of the axial symmetries.

The following picture illustrates the situation in the case $n = 5$:



We write H for the subgroup of G generated by ρ; H is therefore a cyclic group isomorphic to $\mathbb{Z}/n\mathbb{Z}$. Since $\tau\rho\tau^{-1} = \rho^{-1}$, H is actually a normal subgroup of G. Its index is $[G : H] = 2$, so the quotient G/H is isomorphic to $\mathbb{Z}/2\mathbb{Z}$.

We recall that the irreducible representations of H are all of degree 1; they are naturally indexed by $y \in \mathbb{Z}/n\mathbb{Z}$ so that their characters are $\chi_y : \rho^x \mapsto e^{2\pi i x y/n}$. Observe that

$$\mathbb{Z}/n\mathbb{Z} = \{-m, \cdots, -1, 0, 1, 2, \cdots, m\};$$

you may find this useful later.

1. (5 pts) Explain why the conjugacy classes of G are as follows:

   - $\{\mathrm{Id}\}$,
   - $\{\rho, \rho^{-1}\}$,
   - $\{\rho^2, \rho^{-2}\}$,
   - $\vdots$
   - $\{\rho^m, \rho^{-m}\}$,
   - $\{$ All the n axial symmetries $\}$.

2. (10 pts) Use the fact that $G/H \simeq \mathbb{Z}/2\mathbb{Z}$ to write down two irreducible characters of degree 1 of G.

3. Let $y \in \mathbb{Z}/n\mathbb{Z}$, let $\chi_y : \rho^x \mapsto e^{2\pi i x y/n}$ be the corresponding irreducible character of H, and let $\psi_y = \mathrm{Ind}_H^G \chi_y$.

   (a) (20 pts) Write down the values of the character ψ_y.
   You may want to use the identity $e^{it} + e^{-it} = 2\cos t$.

   (b) (10 pts) Determine the decomposition of $\mathrm{Res}_H^G \psi_y$ into irreducible characters of H.

   (c) (25 pts) Use Frobenius reciprocity to determine whether ψ_y is irreducible.
   Your answer may depend on y!

4. (20 pts) Sketch the character table of G.

5. (10 pts) Determine the centre of G and the derived subgroup of G.

This was the only mandatory exercise, that you must submit before the deadline. The following exercise is not mandatory; it is not worth any points, and you do not have to submit it. However, you can try to solve it for practice, and you are welcome to email me if you have questions about them. The solution will be made available with the solution to the mandatory exercises.

Exercise 2 *The character table of D_{2n}, n even*

Redo the same exercise in the case where $n = 2m$ is even.

You may want to consult the previous assignment for inspiration.