Group representations
Exercise sheet 3

https://www.maths.tcd.ie/~mascotn/teaching/2023/MAU34104/index.html

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Email your answers to mascotn@tcd.ie by Thursday March 16, 12:00.

Exercise 1 *An enigmatic group (100 pts)*

As you were walking down the street, you found lying on the ground a finite group G, consisting of 11 conjugacy classes denoted by $C_1, C_2, \cdots, C_{11}$, as well as two (not necessarily irreducible) representations of G whose characters are given by the following data:

Conj. class	C_1	C_2	C_3	C_4	C_5	C_6	C_7	C_8	C_9	C_{10}	C_{11}
Size	1	15	40	90	45	120	144	120	90	15	40
ϕ	6	2	0	0	2	2	1	1	0	-2	3
ψ	21	1	-3	-1	1	1	1	0	-1	-3	0

(the first line lists the conjugacy classes of G; the second line gives the sizes of these conjugacy classes; the last two lines give the values of these two characters).

1. (1 pt) Which conjugacy class does the identity of G lie in?
2. (8 pt) Determine the degree of ϕ, and that of ψ.
3. (1 pt) Determine $\#G$.
4. (30 pts) Is the representation of character ϕ irreducible? If not, determine the shape of its decomposition into a direct sum of irreducible representations, including the degree of each irreducible constituent.
5. (30 pts) Prove that G admits an irreducible representation of degree 16. Write down its character. Finally, give an element of $\mathbb{C}[G]$ which acts as a projector onto the corresponding isotypical component.
6. (30 pts) Prove that G admits a representation of degree 25 which decomposes into the direct sum of 4 pairwise non-isomorphic irreducible representations.

This is the only mandatory exercise, that you must submit before the deadline. The following exercises are not mandatory; they are not worth any points, and you do not have to submit them. However, I *strongly* recommend you try to solve them for practice, and you are welcome to email me if you have questions about them. The solutions will be made available with the solution to the mandatory exercise.

Exercise 2 *Decomposition of permutation representations*

Let G be a finite group acting on a finite set X having at least two elements, and let $g \in G$, $x \in X$. We make the following definitions:

- The *orbit* of x is $G \cdot x = \{h \cdot x,\ h \in G\} \subseteq X$.
- The *fixed set* of g is $\operatorname{Fix} g = \{x \in X \mid g \cdot x = x\} \subseteq X$.

Observe that the orbits form a partition (= disjoint union) of X. In this exercise, we *admit* Burnside's formula, which states that the number of orbits is

$$\frac{1}{\#G} \sum_{g \in G} \# \operatorname{Fix} g.$$

We say that the action of G on X is *transitive* if there is only one orbit, i.e. if for all $x, y \in X$ there exists $g \in G$ such that $g \cdot x = y$. We say that it is *doubly transitive* if for all $x, x', y, y' \in X$ such that $x \neq x'$ and $y \neq y'$, there exists $g \in G$ which achieves simultaneously $g \cdot x = y$ and $g \cdot x' = y'$ (the conditions $x \neq x'$ and $y \neq y'$ are there because this obviously would not be possible if $x = x'$ but $y \neq y'$).

Finally, let π be the permutation representation corresponding to $G \circlearrowright X$. We *admit* (cf. the first lecture of chapter 4) that its character is given by $\chi_\pi(g) = \# \operatorname{Fix} g$ for all $g \in G$.

1. Prove that there exists a representation ρ of G such that $\pi \simeq \mathbb{1} \oplus \rho$.

2. Prove that ρ has no subrepresentation isomorphic to $\mathbb{1}$ iff. the action of G on X is transitive.

3. Suppose that the action of G on X is transitive. Prove that ρ is irreducible iff. the action of G on X is actually doubly transitive.

   Hint: It may help to define an action of G on the set $X \times X$ of (ordered) pairs of elements of X by the formula $g \cdot (x, y) = (g \cdot x, g \cdot y)$. Note that this action cannot be transitive, since the diagonal

   $$\Delta = \{(x, x) \mid x \in X\} \subsetneq X \times X$$

   is clearly stable under G.