Group representations
Exercise sheet 2

`https://www.maths.tcd.ie/~mascotn/teaching/2023/MAU34104/index.html`

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THIS ASSIGNMENT IS NOT MANDATORY.

Email your answers to mascotn@tcd.ie by Friday Feburary 24, 13:00.

Exercise 1 *Preservation of semi-simplicity (50 pts)*

In this exercise, all modules are over a fixed ring R, and all modules are Artinian[1], meaning that there cannot exist an infinite descending chain of submodules.

1. (20 pts) Prove that a submodule of a semi-simple module is also semi-simple.
2. (20 pts) Prove that if $f : M \longrightarrow N$ is a module morphism, and if M is semi-simple, then $\operatorname{Im} f$ is also semi-simple.
3. (10 pts) Let now G be a group, K a field, and $f : V \longrightarrow W$ be a morphism between representations of G over K of finite degree. Prove that if V is semi-simple, then so are $\operatorname{Ker} f$ and $\operatorname{Im} f$.

Exercise 2 *A non-semi-simple ring (50 pts)*

Let K be a field, and let G be a finite group of order $n = \#G$. We will sometimes see the group ring $K[G] = \{\sum_{g \in G} \lambda_g e_g \mid \lambda_g \in K \text{ for all } g \in G\}$ as a module over itself.

1. (5 pts) Let $\Sigma = \sum_{g \in G} e_g \in K[G]$. Prove that $e_h \Sigma = \Sigma$ for all $h \in G$.
2. (5 pts) Prove that $S = \{\lambda\Sigma,\ \lambda \in K\}$ is a sub-$K[G]$-module of $K[G]$.
3. (5 pts) Identify S as a representation of G.

From now on, we assume that $n = 0$ in K.

4. (5pts) Prove that $\Sigma^2 = 0$ in $K[G]$.
5. (10pts) Deduce that $1 - \lambda\Sigma$ is invertible in $K[G]$ for all $\lambda \in K$, where $1 = e_{1_G}$ is the multiplicative identity of $K[G]$.

   Note that since $K[G]$ is not commutative in general, you must prove that your inverse works on both sides.

   Hint: For $x \in \mathbb{R}$ and $m \in \mathbb{N}$, what is the formula for the geometric series $1 + x + x^2 + \cdots + x^m$? How do you prove it?
6. (20 pts) Deduce that $K[G]$, viewed as a $K[G]$-module, is not semi-simple.

[1] NB we make this assumption to make our lives easier, but it can be shown that the properties established in this exercise actually remain valid without this assumption.

These were the only mandatory exercises, that you must submit before the deadline. The following exercise is not mandatory; it is not worth any points, and you do not have to submit them. However, you can try to solve it for practice, and you are welcome to email me if you have questions about them. The solution will be made available with the solution to the mandatory exercises.

Exercise 3 *Annihilators and simple modules*

Let R be a ring, which need not be commutative. We say that $I \subseteq R$ is a *left ideal* if it is an additive subgroup of $(R, +)$ and if $ri \in I$ for all $r \in R$ and $i \in I$. We define *right ideals* similarly. If I is both a left ideal and a right ideal, then we say that it is a *two-sided* ideal. A *maximal* left ideal is a left ideal $M \neq R$ such that there are no left ideals I such that $M \subsetneq I \subsetneq R$.

1. Let M be an R-module. Define its *annihilator* as

$$\operatorname{Ann} M = \{r \in R \mid rm = 0 \text{ for all } m \in M\} \subseteq R.$$

   Prove that $\operatorname{Ann} M$ is a *two-sided* ideal of R.

2. The ring R can be viewed as a module over itself; we denote this module by ${}_R R$, so as to clearly distinguish between the ring R and the R-module ${}_R R$. Identify the submodules of ${}_R R$.

3. Let S be a module, and let $s \in S$.

   (a) Prove that the map

$$\begin{array}{rcl} f_s : {}_R R & \longrightarrow & S \\ r & \longmapsto & rs \end{array}$$

   is a module morphism.

   (b) Prove that S is simple iff. f_s is surjective for all $s \neq 0$.

   (c) Deduce that if S is simple, then it is isomorphic to ${}_R R/M$, where M is a maximal left ideal of R.

   (d) Prove that conversely, if M is a maximal left ideal of R, then ${}_R R/M$ is a simple R-module.

4. Beware that in general, the annihilator of ${}_R R/M$, which is a two-sided ideal, does not agree with the left ideal M! Here is an example: Take $R = M_2(\mathbb{R})$, and $S = \mathbb{R}^2$, which is an R-module if we view its elements as column vectors. Prove that S is simple, determine $\operatorname{Ann} S$, and find a maximal left ideal M of R such that $S \simeq {}_R R/M$.