Group representations
Exercise sheet 1

https://www.maths.tcd.ie/~mascotn/teaching/2023/MAU34104/index.html

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Email your answers to mascotn@tcd.ie by Friday Feburary 10, 13:00.

Exercise 1 *Decomposition over $\mathbb{R}$ (100 pts)*

You are allowed to use without proof the results from Exercises 2-4 to solve this exercise (but I highly recommend you try to solve Exercises 2-4 as well!).

In this exercise, we consider over $K = \mathbb{R}$ two representations of $G = S_3$ constructed in the lectures, namely the permutation representation

$$\mathrm{Perm} : S_3 \longrightarrow \mathrm{GL}_3(K)$$

induced by $S_3 \circlearrowleft \{1, 2, 3\}$, and

$$\triangleleft : S_3 \longrightarrow \mathrm{GL}_2(K), \quad (123) \mapsto \begin{pmatrix} -1 & -1 \\ 1 & 0 \end{pmatrix}, \quad (12) \mapsto \begin{pmatrix} 1 & 1 \\ 0 & -1 \end{pmatrix}$$

obtained by labelling the vertices of an equilateral triangle by $\{1, 2, 3\}$.

1. (35 pts) Prove that $\triangleleft$ is irreducible. Is $\triangleleft$ indecomposable?
2. (65 pts) Prove that $\mathrm{Perm} \simeq \mathbb{1} \oplus \triangleleft$ as representations of S_3.

This is the only mandatory exercise, that you must submit before the deadline. The following exercises are not mandatory; they are not worth any points, and you do not have to submit them. However, I *strongly* recommend you try to solve them for practice, and you are welcome to email me if you have questions about them. The solutions will be made available with the solution to the mandatory exercise.

Exercise 2 *Representation morphisms form a subspace*

Let G be a group, K a field, and let $\rho_1 : G \longrightarrow \mathrm{GL}(V_1)$ and $\rho_2 : G \longrightarrow \mathrm{GL}(V_2)$ be two representations of G over K.

Recall that the set $\mathrm{Hom}(V_1, V_2)$ of all linear transformations from V_1 to V_2 has a vector space structure, with addition and scalar multiplications defined pointwise (that is to say if $T, U \in \mathrm{Hom}_(V_1, V_2)$ and $\lambda \in K$, then $T + U$ is defined as $(T + U)(v_1) = T(v_1) + U(v_1)$ for all $v_1 \in V_1$, and λT is defined as $(\lambda T)(v_1) = \lambda\big(T(v_1)\big)$ for all $v_1 \in V_1$).

Prove that the subset $\mathrm{Hom}_G(V_1, V_2)$ consisting of linear transformations which are representation morphisms is a subspace of $\mathrm{Hom}(V_1, V_2)$.

Exercise 3 *Representations of degree 1*

Prove that a representation of degree 1 is always indecomposable and irreducible.

Exercise 4 *Subrepresentations of degree 1*

Let G be a group, K be a field, and V a representation of G over K. Prove that there exists a subrepresentation of V of degree 1 iff. there exists a nonzero vector $v \in V$ which is an eigenvector for all $g \in G$ (possibly with an eigenvalue which depends on g).

The following exercise has been included for those of you who wish to try their skills in more "exotic" situations. It is not meant to be as profitable for your understanding of the material of this module as the previous exercises. You are still welcome to try to solve it for practice, and to email me if you have questions about it. The solution will be also made available with the solution to the other exercises.

Exercise 5 *Decomposition over $\mathbb{Z}/p\mathbb{Z}$*

Redo Exercise 1, but with $K = \mathbb{Z}/p\mathbb{Z}$ instead of $\mathbb{R}$, with $p \in \mathbb{N}$ prime. Is $\triangleleft$ still irreducible? Indecomposable? Do we still have $\mathrm{Perm} \simeq \mathbb{1} \oplus \triangleleft$?
Your answers may depend on the value of p; explore all cases!