Group representations cheat sheet

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Definition. Representation of a group, subrepresentation, irreducible representation. Degree = dimension. (Iso)morphism of representations. Direct sum of representations.

Theorem (Maschke)**.** *Over $\mathbb{C}$, every representation of a finite group decomposes into a direct sum of irreducible representations.*

Definition. Character: $\chi(g) = \operatorname{Tr} \rho(g)$.

Definition. Trivial character: $\mathbb{1}(g) = 1$ for all $g \in G$.

Proposition. $\chi(1_g) = \deg \chi$ *(meaning* $\dim \chi$*).*

Lemma. *Let V_1, V_2 have characters χ_1, χ_2. Then $V_1 \oplus V_2$ has character $\chi_1 + \chi_2$,* $\operatorname{Hom}(V_1, V_2)$ *has character $\overline{\chi_1}\chi_2$, and* $\operatorname{Hom}(V_1, \mathbb{C})$ *has character $\overline{\chi_1}$.*

Theorem. *Let ϕ, ψ be characters of G. Then $\phi + \psi$, $\overline{\phi}$, and $\phi\psi$ are characters of G. If* $\deg \phi = 1$*, then ϕ is irreducible. If ϕ is irreducible, so is $\overline{\phi}$. If ϕ is irreducible and* $\deg \psi = 1$*, then $\phi\psi$ is irreducible.*

Theorem. *Two representations are isomorphic $\iff$ they have the same character.*

Definition. Inner product of characters: $(\phi \mid \psi) = \dfrac{1}{\#G} \sum_{g \in G} \phi(g)\overline{\psi(g)}$.

Lemma. $(\phi \mid \psi) = \dim \operatorname{Hom}_G(\phi, \psi)$ *is always a nonnegative integer.*

Theorem. *# irreducible representations of G = # conjugacy classes of G.*
This means that the character table is ***square****.*

Let $W_1, \cdots, W_r$ be the irreducible representations of G. Let $\chi_1, \cdots, \chi_r$ be their characters, and let $d_1, \cdots, d_r$ be their degrees.

Remark. WLOG, $W_1 = \mathbb{1}$, $\chi_1 = \mathbb{1}$, $d_1 = 1$.

Theorem. $d_1^2 + \cdots + d_r^2 = \#G$.

Corollary. *G Abelian $\iff d_1 = \cdots = d_r = 1$.*

Theorem (First orthogonality relations)**.** $(\chi_i \mid \chi_j) = \begin{cases} 1 \text{ if } i = j, \\ 0 \text{ if } i \neq j. \end{cases}$
This expresses that the ***rows*** *of the character table are orthonormal.*

Theorem (Second orthogonality relations)**.** *Let $g, h \in G$, and let $C = \operatorname{cc}_G(g)$ be the conjugacy class of g. Then* $\sum_{j=1}^{r} \chi_j(g)\overline{\chi_j(h)} = \begin{cases} \frac{\#G}{\#C} \text{ if } h \in C, \\ 0 \text{ if } h \notin C. \end{cases}$
This expresses that the ***columns*** *of the character table are orthogonal.*

Theorem (Information about G from its character table)**.** *For each* $j = 1, \cdots, r$, *let* $N_j = \operatorname{Ker} \chi_j = \{g \in G \mid \chi_j(g) = d_j\}$ *(remember* $d_j = \chi_j(1_G)$*). Then the* N_j *are* normal subgroups *of* G*; conversely, every normal subgroup of* G *is obtained as the intersection of some of the* N_j*.*
The derived subgroup of G *is* $D(G) = \bigcap_{d_j=1} N_j$*.*
The centre of G *is the set of elements that sit alone in their own conjugacy class.*

Theorem (Fundamental)**.** *Let* V *be a representation of* G *of character* ψ*.*
Suppose V *decomposes into* $(n_1$ *copies of* $W_1) \oplus \cdots \oplus (n_r$ *copies of* $W_r)$*. Then*

- $\psi = n_1\chi_1 + \cdots + n_r\chi_r$,
- $n_j = (\psi \mid \chi_j)$ *for all* $j = 1, \cdots, r$,
- $(\psi \mid \psi) = n_1^2 + \cdots + n_r^2$*. In particular,*
- V *is irreducible* $\Longleftrightarrow (\psi \mid \psi) = 1$.

Definition. Let $V = \bigoplus_i V_i$ be a decomposition of V into irreducible subrepresentations, and let W be an irreducible representation of G. The W-isotypical component of V is the subrepresentation of V formed by the V_i such that $V_i \simeq W$.

Theorem. *The projection onto the* W_j*-isotypical component parallel to the others is*

$$\pi_j = \frac{d_j}{\# G} \sum_{g \in G} \overline{\chi_j(g)} \rho(g).$$

This means that $\pi_{j|V_i} = \begin{cases} \mathrm{Id} & \text{if } V_i \simeq W_j, \\ 0 & \text{if } V_i \not\simeq W_j. \end{cases}$

Let now H be a subgroup of G. Let ψ be a character of G, and let η be a character of H.

Proposition. $\operatorname{Res}_H^G \psi = \psi_{|H}$.

Theorem. $\deg \operatorname{Ind}_H^G \eta = [G : H] \deg \eta$.

Lemma (First induced character formula, less useful)**.** *For all* $g \in G$,

$$(\operatorname{Ind}_H^G \eta)(g) = \frac{1}{\# H} \sum_{x \in G} \eta^0(xgx^{-1}), \quad \text{where} \quad \eta^0(g) = \begin{cases} \eta(g) & \text{if } g \in H, \\ 0 & \text{if } g \notin H. \end{cases}$$

Theorem (Second induced character formula, more useful)**.** *Let* $g \in G$*. Decompose* $\operatorname{cc}_G(g) \cap H = \coprod_j \operatorname{cc}_H(h_j)$*. Then* $(\operatorname{Ind}_H^G \eta)(g) = \dfrac{[G : H]}{\# \operatorname{cc}_G(g)} \sum_j \# \operatorname{cc}_H(h_j) \eta(h_j)$ *(this means* 0 *if* $\operatorname{cc}_G(g) \cap H = \emptyset$*).*

Theorem (Frobenius reciprocity, even more useful)**.**

$$(\operatorname{Ind}_H^G \eta \mid \psi)_G = (\eta \mid \operatorname{Res}_H^G \psi)_H.$$