Galois theory — Exercise sheet 2

https://www.maths.tcd.ie/~mascotn/teaching/2023/MAU34101/index.html

Version: October 18, 2023

Submit[1] your answers by Monday November 6th, 4PM.

Exercise 1 *From the 2021 exam (100 pts)*

Let $f(x) = x^4 - 5x^2 + 1 \in \mathbb{Q}[x]$. We admit without proof that $f(x)$ is irreducible in $\mathbb{Q}[x]$. Let $\alpha \in \mathbb{C}$ be a root of $f(x)$, and let $K = \mathbb{Q}(\alpha)$.

Note: This is the exercise form the 2021 exam I warned you about in the lectures. So even if it is possible to find an expression for α in terms of nested square roots, you are strongly advised to refrain from doing so, and to merely rely on the relation $f(\alpha) = 0$ instead.

1. (5 pts) Express the complex roots of $f(x)$ in terms of α.

   Hint: Check out $1/\alpha$.

2. (20 pts) Prove that K is a Galois extension of $\mathbb{Q}$.

3. (25 pts) Prove that $\mathrm{Gal}(K/\mathbb{Q}) \simeq (\mathbb{Z}/2\mathbb{Z}) \times (\mathbb{Z}/2\mathbb{Z})$, and explain how its elements act on the conjugates of α.

4. (40 pts) Draw a diagram showing all the intermediate fields $\mathbb{Q} \subseteq E \subseteq K$, and identifying these intermediate fields explicitly. Justify your answer.

5. (10 pts) Prove that $K = \mathbb{Q}(\sqrt{3}, \sqrt{7})$.

This was the only mandatory exercise, that you must submit before the deadline. The following exercises are not mandatory; they are not worth any points, and you do not have to submit them. However, I highly recommend that you try to solve them for practice, and you are welcome to email me if you have questions about them. The solutions will be made available with the solution to the mandatory exercise.

---

[1] Preferably in paper form, or by emailing LaTeXdocuments to mismet@tcd.ie

Exercise 2 *Yes or no?*

Let $f(x) = x^3 + x + 1 \in \mathbb{Q}[x]$ (you may assume without proof that f is irreducible over $\mathbb{Q}$), and let $L = \mathbb{Q}[x]/(f)$.

1. Is L a separable extension of $\mathbb{Q}$? Explain.
2. Is L a normal extension of $\mathbb{Q}$? Explain.

   Hint: What does the fact that $f : \mathbb{R} \longrightarrow \mathbb{R}$ is strictly increasing tell you about the complex roots of f?
3. Is L a Galois extension of $\mathbb{Q}$? Explain.

Exercise 3 *A cyclic biquadratic extension*

Let $\alpha = \sqrt{13}$, $K = \mathbb{Q}(\alpha)$, $\beta = i\sqrt{65 + 18\sqrt{13}}$ (where $i^2 = -1$), $\beta' = i\sqrt{65 - 18\sqrt{13}}$ (note that $65 > 18\sqrt{13}$), and $L = \mathbb{Q}(\beta)$.

1. Prove that the minimal polynomial of β over $\mathbb{Q}$ is

$$M(x) = (x^2 + 65)^2 - 18^2 \cdot 13 = x^4 + 130x^2 + 13.$$

2. What are the Galois conjugates of β over $\mathbb{Q}$?
3. Prove that L is a Galois extension of $\mathbb{Q}$.

   Hint: Check that $\beta\beta' = -\alpha$.
4. Explain why there exists an element $\sigma \in \mathrm{Gal}(L/\mathbb{Q})$ such that $\sigma(\beta) = \beta'$.
5. Let $\sigma \in \mathrm{Gal}(L/\mathbb{Q})$ be such that $\sigma(\beta) = \beta'$ as above. Explain why $\sigma(\alpha)$ makes sense, and determine $\sigma(\alpha)$.
6. Let again $\sigma \in \mathrm{Gal}(L/\mathbb{Q})$ be such that $\sigma(\beta) = \beta'$ as above. Determine the action of σ on the conjugates of β.

   Hint: Again, $\beta\beta' = -\alpha$.
7. Deduce that $\mathrm{Gal}(L/\mathbb{Q}) \simeq \mathbb{Z}/4\mathbb{Z}$.
8. Sketch a diagram showing all the fields $\mathbb{Q} \subseteq E \subseteq L$, ordered by inclusion.
9. Does $i\sqrt{13} \in L$?

Exercise 4 *More square roots*

You may want to use the results established in Exercise 5 of the previous assignment to solve this exercise.

Let $L = \mathbb{Q}(\sqrt{10}, \sqrt{42})$.

1. Prove that L is a Galois extension of $\mathbb{Q}$.
2. Prove that $[L : \mathbb{Q}] = 4$.
3. Describe all the elements of $\mathrm{Gal}(L/\mathbb{Q})$. What is $\mathrm{Gal}(L/\mathbb{Q})$ isomorphic to?
4. Sketch the diagram showing all intermediate extensions $\mathbb{Q} \subseteq E \subseteq L$, ordered by inclusion. Explain clearly which field corresponds to which subgroup.
5. Does $\sqrt{15} \in L$? Use the previous question to answer.

Exercise 5 *The fifth cyclotomic field*

In this exercise, we consider the primitive 5th root $\zeta = e^{2\pi i/5}$, and we set $L = \mathbb{Q}(\zeta)$. We know that L is Galois over $\mathbb{Q}$, so we define $G = \mathrm{Gal}(L/\mathbb{Q})$. We also let

$$c = \frac{\zeta + \zeta^{-1}}{2} = \cos(2\pi/5) = 0.309\cdots,$$

$$C = \mathbb{Q}(c),$$

and finally

$$c' = \frac{\zeta^2 + \zeta^{-2}}{2} = \cos(4\pi/5) = -0.809\cdots.$$

1. Write down explicitly the minimal polynomial of ζ over $\mathbb{Q}$, and express its complex roots in terms of ζ.
2. Deduce that $\zeta + \zeta^2 + \zeta^3 + \zeta^4 = -1$.
3. Prove that G is a cyclic group. What is its order? Find an explicit generator of G.
4. Deduce that $c \notin \mathbb{Q}$.
5. Make the list of all subgroups of G.
6. Draw a diagram showing all the fields E such that $\mathbb{Q} \subset E \subset L$, ordered by inclusion.
7. What are the conjugates of c over $\mathbb{Q}$? Determine explicitly the minimal polynomial of c over $\mathbb{Q}$ (exact computations only, computations with the approximate value of c are forbidden).
8. Deduce that
$$c = \frac{-1 + \sqrt{5}}{4}.$$

9. What are the conjugates of ζ over C (as opposed to over $\mathbb{Q}$)?

10. Deduce that
$$\zeta = \frac{-1+\sqrt{5}+i\sqrt{10+2\sqrt{5}}}{4}.$$

Exercise 6 *Bioche vs. Galois*

This exercise has an unusual flavour, and is not representative of what you should expect for the final exam. The goal of this exercise is to give a Galois-theoretic interpretation of Bioche's rules (cf. `https://en.wikipedia.org/wiki/Bioche%27s_rules`*), which are rules suggesting appropriate substitutions to turn integrals involving trigonometric functions into integrals of rational fractions. Knowledge of Bioche's rules is not required to solve this exercise.*

In this exercise, we use the shorthands s for the sine function and c for the cosine function, and we denote by $\mathbb{C}(s,c)$ the set of rational fractions in $\sin x$ and $\cos x$ with complex coefficients, meaning of expressions such as
$$\frac{2sc^3-i}{c-7s+3} = \frac{2\sin x\cos^3 x - i}{\cos x - 7\sin x + 3}.$$
Observe that $\mathbb{C}(s,c)$ is a field with respect to point-wise addition and multiplication.

We write $\mathbb{C}(c)$ for the subfield of $\mathbb{C}(s,c)$ consisting of rational fractions which can be expressed in terms of c only, and similarly $\mathbb{C}(s)$ for rational fractions in s only. For example, $\frac{c^3-2c^2+2i}{ic-1} \in \mathbb{C}(c)$, but $s \notin \mathbb{C}(c)$ since all the elements of $\mathbb{C}(c)$ are even functions whereas s is not; observe however that $s^2 \in \mathbb{C}(c)$ since $s^2 = 1-c^2$.

We also define $K = \mathbb{C}(s) \cap \mathbb{C}(c) \subset \mathbb{C}(s,c)$, so that for instance the function $c_2 = \cos(2x)$ lies in K since $c_2 = 2c^2-1 = 1-2s^2$.

Finally, we define
$$\begin{array}{rcl} \mu : \mathbb{C}(s,c) & \to & \mathbb{C}(s,c) \\ f(x) & \mapsto & f(-x), \end{array} \qquad \begin{array}{rcl} \tau : \mathbb{C}(s,c) & \to & \mathbb{C}(s,c) \\ f(x) & \mapsto & f(x+\pi), \end{array} \qquad \begin{array}{rcl} \sigma : \mathbb{C}(s,c) & \to & \mathbb{C}(s,c) \\ f(x) & \mapsto & f(\pi-x); \end{array}$$
observe that these are field automorphisms of $\mathbb{C}(s,c)$ which are involutive and commute with each other, so they generate the subgroup
$$G = \{\mathrm{Id},\, \mu=\sigma\tau,\, \tau=\mu\sigma,\, \sigma=\mu\tau\} \simeq (\mathbb{Z}/2\mathbb{Z}) \times (\mathbb{Z}/2\mathbb{Z})$$
of $\mathrm{Aut}\big(\mathbb{C}(s,c)\big)$.

1. Prove that the four inclusions $K \subset \mathbb{C}(s) \subset \mathbb{C}(s,c)$ and $K \subset \mathbb{C}(s) \subset \mathbb{C}(s,c)$ are all strict.

2. Prove that $[\mathbb{C}(s):K] = [\mathbb{C}(s,c):\mathbb{C}(s)] = [\mathbb{C}(c),K] = [\mathbb{C}(s,c):\mathbb{C}(c)] = 2$.

3. Prove that $K = \mathbb{C}(c_2)$, where $\mathbb{C}(c_2)$ is the field of rational fractions expressible in terms of c_2 only.

4. Prove that the extension $\mathbb{C}(s,c)/K$ is Galois, and describe its Galois group.

5. Let $f \in \mathbb{C}(s,c)$. Prove that if f is invariant by any two of μ, τ, σ, then it is also invariant by the third one, and that in this case $f \in \mathbb{C}(c_2)$.

6. Determine the minimal polynomials over K of the elements $t = \tan x = s/c$ and $s_2 = \sin(2x) = 2sc$ of $\mathbb{C}(s, c)$.

7. Draw a diagram showing all the subgroups of $\mathrm{Gal}(\mathbb{C}(s, c)/K)$.

8. Draw a diagram showing all the intermediate fields E between K and $\mathbb{C}(s, c)$. Where are the fields $\mathbb{C}(t)$, $\mathbb{C}(s_2, c_2)$, and $\mathbb{C}(s_2)$ on this diagram?

Make sure find an explanation for all the surprising conclusions you may be led to!