Introduction to number theory
Exercise sheet 5

https://www.maths.tcd.ie/~mascotn/teaching/2021/MAU22301/index.html

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Email your answers to makindeo@tcd.ie by Wednesday December 1st, 2PM.
The use of electronic calculators and computer algebra software is allowed.

Exercise 1 *A Pell-Fermat equation (100 pts)*

In this Question, we consider the Diophantine equation

$$x^2 - 33y^2 = 1, \quad x, y \in \mathbb{Z} \quad (\star).$$

1. (20 pts) Compute the continued fraction of $x = \sqrt{33}$.

   This means that you must somehow find a formula for all *the coefficients a_n, $n \in \mathbb{Z}_{\geq 0}$.*

2. (20 pts) Compute the convergents of the continued fraction of $\sqrt{33}$ for $n \leqslant 4$.
3. (10 pts) Find a nontrivial ($y \neq 0$) solution to the equation ($\star$).
4. (20 pts) Find a solution to the equation ($\star$) such that $x \geqslant 1000$.
5. (10 pts) What is the fundamental unit in $\mathbb{Z}[\sqrt{33}]$?
6. (20 pts) Does the Diophantine equation $x^2 - 33y^2 = -1$ have solutions $x, y \in \mathbb{Z}$?

   Hint: Use the previous question.

Solution 1

1. Since $\sqrt{33}$ is a quadratic irrational, the coefficients a_n are ultimately periodic. We compute x_n, a_n by the relations $x_0 = x$, $a_n = \lfloor x_n \rfloor$, and $x_{n+1} = \frac{1}{x_n - a_n}$. As $5 < \sqrt{33} < 6$, we find

n	x_n	a_n
0	$\sqrt{33}$	5
1	$\frac{1}{\sqrt{33}-5} = \frac{5+\sqrt{33}}{(\sqrt{33}-5)(\sqrt{33}+5)} = \frac{5+\sqrt{33}}{8}$	1
2	$\frac{1}{\frac{5+\sqrt{33}}{8}-1} = \frac{8}{-3+\sqrt{33}} = \frac{8(3+\sqrt{33})}{(-3+\sqrt{33})(3+\sqrt{33})} = \frac{3+\sqrt{33}}{3}$	2
3	$\frac{1}{\frac{3+\sqrt{33}}{3}-2} = \frac{3}{-3+\sqrt{33}} = \frac{3(3+\sqrt{33})}{(-3+\sqrt{33})(3+\sqrt{33})} = \frac{3+\sqrt{33}}{8}$	1
4	$\frac{1}{\frac{3+\sqrt{33}}{8}-1} = \frac{8}{-5+\sqrt{33}} = \frac{8(5+\sqrt{33})}{(-5+\sqrt{33})(5+\sqrt{33})} = 5+\sqrt{33}$	10
5	$\frac{1}{\frac{5+\sqrt{33}}{3}-10} = \frac{1}{\frac{\sqrt{33}}{3}-5} = x_1.$	

Since $x_5 = x_1$, we also have $a_5 = a_1$, so the process is periodic from ow on. In conclusion,
$$\sqrt{33} = [5, \overline{1, 2, 1, 10}].$$

2. We compute p_n and q_n for $n \leqslant 4$ by the relations $p_{-2} = 0$, $p_{-1} = 1$, $q_{-2} = 1$, $q_{-1} = 0$, $p_n = a_n p_{n-1} + p_{n-2}$, $q_n = a_n q_{n-1} + q_{n-2}$; and we know that the n-th convergent is p_n/q_n. We find

n	a_n	p_n	q_n
-2		0	1
-1		1	0
0	5	5	1
1	1	6	1
2	2	17	3
3	1	23	4
4	10	247	43.

3. We begin by looking for $n \geq 0$ such that $p_n^2 - 33q_n^2 = \pm 1$. The first time this happens is for $n = 3$, which yields the solution $x = 23$, $y = 4$.

4. The solution found in the previous question corresponds to the unit
$$u = 23 + 4\sqrt{33} \in \mathbb{Z}[\sqrt{33}]^\times$$
of norm 1. In order to get larger solutions, we consider its powers u^k. Already for $k = 2$, we find
$$u^2 = (23 + 4\sqrt{33})^2 = 23^2 + 4^2 \cdot 33 + 2 \cdot 23 \cdot 4\sqrt{33} = 1057 + 184\sqrt{33},$$
whence the solution $x = 1057$, $y = 184$.

5. Since we saw two questions ago that u corresponds to the smallest non-trivial solution, it is a fundamental unit.

6. No. Indeed, any solution would correspond to a unit of norm -1; yet by Dirichlet's theorem and the previous question, we have
$$\mathbb{Z}[\sqrt{33}]^\times = \{\pm u^k \mid k \in \mathbb{Z}\},$$
and since $N(\pm u^k) = N(\pm 1)N(u)^k = +1$ for all k by multiplicativity of the norm, we conclude that all the units have norm $+1$, so none has norm -1.

This was the only mandatory exercise, that you must submit before the deadline. The following exercises are not mandatory; they are not worth any points, and you do not have to submit them. However, I highly recommend that you try to solve them for practice, and you are welcome to email me if you have questions about them. The solutions will be made available with the solution to the mandatory exercise.

Exercise 2 *The battle of Hastings*

The battle of Hastings, which took place on October 14, 1066, was a major battle in History.

The following fictional historical text, taken from *Amusement in Mathematics* (H. E. Dundeney, 1917), refers to it:

"The men of Harold stood well together, as their wont was, and formed thirteen squares, with a like number of men in every square thereof. (...) When Harold threw himself into the fray the Saxons were one mighty square of men, shouting the battle cries 'Ut!', 'Olicrosse!', 'Godemite!'."

Use continued fractions to determine how many soldiers this fictional historical text suggests Harold II had at the battle of Hastings.

Solution 2

We are looking for solutions to $13y^2 + 1 = x^2$ with $x, y \in \mathbb{Z}_{\geqslant 0}$. This translates into the Pell-Fermat equation $x^2 - 13y^2 = 1$.

Clearly, the trivial solution $x = 1$, $y = 0$ does not reflect the situation (I doubt Harold II would have gone to battle alone !), so let us compute the continued fraction expansion of $x = \sqrt{13}$ until we find a non-trivial solution.

$$
\begin{aligned}
x_0 &= \sqrt{13}, & a_0 &= \lfloor x_0 \rfloor = 3, & p_0 &= 3, q_0 = 1, & p_0^2 - 13q_0^2 &= -4 \neq \pm 1. \\
x_1 = \frac{1}{\sqrt{13}-3} &= \frac{3+\sqrt{13}}{4}, & a_1 &= \lfloor x_1 \rfloor = 1, & p_1 &= 4, q_1 = 1, & p_1^2 - 13q_1^2 &= 3 \neq \pm 1. \\
x_2 = \frac{1}{\frac{3+\sqrt{13}}{4}-1} &= \frac{1+\sqrt{13}}{3}, & a_2 &= \lfloor x_2 \rfloor = 1, & p_2 &= 7, q_2 = 2, & p_2^2 - 13q_2^2 &= -3 \neq \pm 1. \\
x_3 = \frac{1}{\frac{1+\sqrt{13}}{3}-1} &= \frac{2+\sqrt{13}}{3}, & a_3 &= \lfloor x_3 \rfloor = 1, & p_3 &= 11, q_3 = 3, & p_3^2 - 13q_3^2 &= 4 \neq \pm 1. \\
x_4 = \frac{1}{\frac{2+\sqrt{13}}{3}-1} &= \frac{1+\sqrt{13}}{4}, & a_4 &= \lfloor x_4 \rfloor = 1, & p_4 &= 18, q_4 = 5, & p_4^2 - 13q_4^2 &= -1.
\end{aligned}
$$

We have found the fundamental unit $\varepsilon = 18 + 5\sqrt{13}$ of norm $N(\varepsilon) = -1$. We deduce that the fundamental solution to our equation corresponds to

$$\varepsilon^2 = 649 + 180\sqrt{13},$$

that is to say $x = 649$, $y = 180$.

Since the other solutions, which correspond to powers of ε^2, are even larger, this suggests a number of soldiers on this side of the battle (including Harold II) was at least $649^2 = 421201$. That's really a lot, which confirms that this text is certainly fictional!

Exercise 3 *Continued fraction vs. series*

Let $x \in (0,1)$ be irrational, and let $[a_0, a_1, \cdots, a_n] = p_n/q_n$ ($n \in \mathbb{Z}_{\geq 0}$) be the convergents of the continued fraction expansion of x. Prove that

$$x = \sum_{n=0}^{+\infty} \frac{(-1)^n}{q_n q_{n+1}}.$$

Hint: Where could the $(-1)^n$ *come from ?*

Solution 3

We know that $q_n p_{n-1} - p_n q_{n-1} = (-1)^n$ for all n. Therefore, we have

$$\frac{p_n}{q_n} - \frac{p_{n-1}}{q_{n-1}} = \frac{(-1)^{n-1}}{q_n q_{n-1}}$$

for all n. Now, obviously

$$\frac{p_m}{q_m} = \left(\frac{p_m}{q_m} - \frac{p_{m-1}}{q_{m-1}}\right) + \left(\frac{p_{m-1}}{q_{m-1}} - \frac{p_{m-2}}{q_{m-2}}\right) + \cdots + \left(\frac{p_1}{q_1} - \frac{p_0}{q_0}\right) \frac{p_0}{q_0} = \frac{p_0}{q_0} + \sum_{n=1}^{m} \left(\frac{p_n}{q_n} - \frac{p_{n-1}}{q_{n-1}}\right).$$

On the one hand, $p_0 = a_0 = \lfloor x \rfloor = 0$ since $x \in (0,1)$, so $\frac{p_0}{q_0} = 0$. On the other hand, we know that the sequence $\frac{p_n}{q_n}$ converges to x, so we get

$$x = \lim_{m\to\infty} \frac{p_m}{q_m} = \lim_{m\to\infty} \sum_{n=1}^{m} \left(\frac{p_n}{q_n} - \frac{p_{n-1}}{q_{n-1}}\right) = \sum_{n=1}^{+\infty} \left(\frac{p_n}{q_n} - \frac{p_{n-1}}{q_{n-1}}\right) = \sum_{n=1}^{+\infty} \frac{(-1)^{n-1}}{q_n q_{n-1}} = \sum_{n=0}^{+\infty} \frac{(-1)^n}{q_n q_{n+1}}.$$