# Rings, fields, and modules
# Exercise sheet 3

https://www.maths.tcd.ie/~mascotn/teaching/2021/MAU22102/index.html

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Email your answers to aylwarde@tcd.ie by Monday March 22, 4PM.

## **Exercise 1** *Associates to irreducibles (20 pts)*

Let $R$ be a commutative ring, which is not necessarily a domain.

1. (6 pts) Let $x, y \in R$. Prove that if $xy \in R^\times$, then $x \in R^\times$ and $y \in R^\times$.
2. (14 pts) Let $p \in R$ be irreducible, and let $x \in R$. Prove that if $x$ is associate to $p$, then $x$ is also irreducible.

## Solution 1

1. Suppose that $xy \in R^\times$. Then we can find $z \in R$ such that $xyz = 1$. Therefore $x(yz) = 1$ so $x \in R^\times$, and $y(xz) = 1$ so $y \in R^\times$ as well.
2. Suppose that $p$ and $x$ are associates. Then there exist $a, b \in R$ such that $p = ax$ and $x = bp$. This implies $p = abp = (ab)p$; as $p$ is irreducible, either $p \in R^\times$ or $ab \in R^\times$. But $p \notin R^\times$, since $p$ is irreducible; therefore $ab \in R^\times$, whence $a \in R^\times$ and $b \in R^\times$ as well. From this, we conclude that $x \neq 0$ (else $p = ax = 0$, absurd), and that $x \notin R^\times$ (else $p = ax \in R^\times$, absurd).

   Suppose now that we have a factorisation $x = yz$ with $y, z \in R$. Then $p = ax = ayz = (ay)z$. Since $p$ is irreducible, either $ay \in R^\times$ whence $y \in R^\times$ by the previous question, or $z \in R^\times$.

   In conclusion, $x \neq 0$, $x \notin R^\times$, and $x = yz$ implies that $y \in R^\times$ or $z \in R^\times$: this is the textbook definition of $x$ being irreducible.

## **Exercise 2** *Nilpotent elements (80 pts)*

Let $R$ be a commutative ring. We say that an element $x \in R$ is *nilpotent* if there exists an integer $n \geqslant 1$ such that $x^n = 0$. We write $\mathrm{Nil}(R) \subset R$ for the subset of $R$ formed of the nilpotent elements of $R$.
Example: if $R = \mathbb{Z}/8\mathbb{Z}$, then $x = \bar{2} \in \mathrm{Nil}(R)$, since $x^3 = \bar{8} = \bar{0}$ in $R$; in fact

$$\mathrm{Nil}(\mathbb{Z}/8\mathbb{Z}) = \{\bar{0}, \bar{2}, \bar{4}, \bar{6}\}.$$

Note that by definition, the 0 of $R$ lies in $\mathrm{Nil}(R)$.

1. (5 pts) Prove that if $R$ is a domain, then $\mathrm{Nil}(R)$ is reduced to $\{0\}$.

2. (5 pts) Prove that if $x \in \mathrm{Nil}(R)$, then $1 - x \in R^\times$.

   *Hint: What is* $(1-x)(1+x+x^2+\cdots+x^n)$*?*

3. (10 pts) Let $x, y \in R$. Prove that if $x$ and $y$ are both nilpotent, then so is $x+y$.

   *Hint: Use the binomial formula to expand* $(x+y)^n$ *for some large enough* $n \in \mathbb{N}$.

4. (5 pts) Prove that $\mathrm{Nil}(R)$ is an ideal of $R$.

5. (15 pts) Prove that the quotient ring $R/\mathrm{Nil}(R)$ has no nonzero nilpotents, i.e. that $\mathrm{Nil}(R/\mathrm{Nil}(R)) = \{\bar{0}\}$ where $\bar{0}$ denotes the 0 of the quotient $R/\mathrm{Nil}(R)$.

6. (15 pts) Prove that $\mathrm{Nil}(R)$ is contained in the intersection of all prime ideals of $R$.

   *Hint: Let* $P \subset R$ *be a prime ideal. Which property does the quotient ring* $R/P$ *have? How does this relate to this exercise?*

   From now on, we admit without proof the fact that actually,

$$\mathrm{Nil}(R) = \bigcap_{P \text{ prime ideal } \lhd R} P$$

   agrees with the intersection of all prime ideals of $R$.

7. (25 pts) Let $F(x) \in R[x]$. Prove that if $F(x)$ is invertible in $R[x]$ iff. its constant coefficient is invertible in $R$ and its other coefficients are all nilpotent.

   *Hint: Use some of the previous questions. What would* $R[x]^\times$ *be if* $R$ *were a domain?*

## Solution 2

1. Let $x \in \mathrm{Nil}(R)$. Then $0 = x^n = x \cdots x$ for some $n \geqslant 1$. Since $R$ is assumed to be a domain, this forces one of the factors to be 0, whence $x = 0$.

2. Since $x \in \mathrm{Nil}(R)$, there exists $n \geqslant 1$ such that $x^n = 0$. Expanding $(1-x)(1+x+x^2+\cdots+x^{n-1})$, we get $1-x+x-x^2+x^2+\cdots-x^{n-1}+x^{n-1}-x^n = 1-x^n = 1$. This shows that $1-x$ is invertible, with inverse $1+x+\cdots+x^{n-1}$.

3. Since $x \in \mathrm{Nil}(R)$, there exists $n \geqslant 1$ such that $x^n = 0$. Similarly, there exists $m \geqslant 1$ such that $y^m = 0$. Then, by the binomial formula (valid since $R$ is commutative), we have

$$(x+y)^{m+n} = \sum_{k=0}^{n+m} \binom{n+m}{k} x^k y^{n+m-k}.$$

   The terms for $k \leqslant n$ are divisible by $y^m$, so they vanish, whereas those with $k \geqslant n$ are divisible by $x^n$, so they also vanish. Thus all terms are 0, so $(x+y)^{n+m} = 0$, which shows that $x + y \in \mathrm{Nil}(R)$.

4. We already know (by definition) that $0 \in \operatorname{Nil}(R)$, and we have just proved that $\operatorname{Nil}(R)$ is closed by sum. It remains to show that is is also closed by multiplication by $R$. Let $x \in \operatorname{Nil}(R)$, so there exists $n \geqslant 1$ such that $x^n = 0$, and let $y \in \mathbb{R}$. (Since $R$ is commutative) we have $(xy)^n = x^n y^n = 0y^n = 0$ so $xy \in \operatorname{Nil}(R)$, as wanted. This shows that $\operatorname{Nil}(R)$ is an ideal of $R$.

   *Remark: The commutativity assumption is crucial here. For instance, in the ring of $n \times n$ matrices over $\mathbb{R}$, a matrix is a product of nilpotents iff. it is not invertible, so in particular the (two-sided) ideal generated by the nilpotents is the whole ring.*

5. Let $\bar{x} \in \operatorname{Nil}(R/\operatorname{Nil}(R))$, so there exists $n \geqslant 1$ such that $\bar{x}^n = \bar{0}$. This means that if $x \in R$ is any representative of $R$, then $x^n \in \operatorname{Nil}(R)$. Therefore, there exists $m \geqslant 1$ such that $(x^n)^m = 0$. Thus $x^{nm} = (x^n)^m = 0$, so $x \in \operatorname{Nil}(R)$. But this means that $\bar{x} = \bar{0}$ in $R/\operatorname{Nil}(R)$.

6. Let $P \subset R$ be a prime ideal, and let $x \in \operatorname{Nil}(R)$; we want to show that $x \in P$. Since $x \in \operatorname{Nil}(R)$, there exists $n \geqslant 1$ such that $x^n = 0$; thus $\bar{x}^n = \bar{0}$ in the quotient $R/P$ as well, whence $\bar{x} \in \operatorname{Nil}(R/P)$. But since $P$ is prime, $R/P$ is a domain, so by the first question we have $\operatorname{Nil}(R/P) = \{\bar{0}\}$, whence $\bar{x} = 0$. This means that $x \in P$.

7. Write $F(x) = f_d x^d + \cdots + f_1 x + f_0$. If $f_0$ is invertible, then we can write $F(x) = f_0(1 - F_1(x))$ where $F_1(x) = -f_0^{-1}(f_d x^d + \cdots + f_1 x)$. So if $f_i$ is nilpotent for all $i > 0$, then as the set of nilpotents of $R[x]$ forms an ideal by Question 4, the polynomial $F_1(x)$ is also nilpotent, so $1 - F_1(x)$ is invertible; therefore so is $f_0(1 - F_1(x))$, with inverse $f_0^{-1}(1 + F_1(x) + F_1(x)^2 + \cdots)$.

   Conversely, suppose that $F(x)$ is invertible, with inverse $G(x) \in R[x]$. Evaluating $F(x)G(x) = 1$ at $x = 0$, we see that the constant coefficient of $F$ is invertible, with inverse the constant coefficient of $G$. Besides, let $P \lhd R$ be a prime ideal; we still have $\overline{F(x)G(x)} = \overline{1}$ in $(R/P)[x]$, but since $P$ is prime $R/P$ is a domain, so there we must have $\deg \overline{F} + \deg \overline{G} = \deg \overline{F}\,\overline{G} = \deg \overline{1} = 0$, whence $\deg \overline{F} = 0$, which means that all the non-constant coefficients of $F$ lie in $P$. Since this holds for all $P$, we conclude that these coefficients lie in $\bigcap_P P = \operatorname{Nil}(R)$.