Rings, fields, and modules
Exercise sheet 3

https://www.maths.tcd.ie/~mascotn/teaching/2021/MAU22102/index.html

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Email your answers to aylwarde@tcd.ie by Monday April 5, 4PM.

Exercise 1 *An irreducible polynomial of degree 4 (100 pts)*

Let $f(x) = x^4 + x^2 - 2x - 1 \in \mathbb{Z}[x]$.

1. (2 pts) Briefly explain why $\mathbb{Z}/2\mathbb{Z}$ and $\mathbb{Z}/3\mathbb{Z}$ are fields.

2. (15 pts) Determine the factorisation of f in $(\mathbb{Z}/3\mathbb{Z})[x]$. Do not forget to prove that your factorisation is complete!

   Hint: Which of the elements of $\mathbb{Z}/3\mathbb{Z} = \{\overline{0}, \overline{1}, \overline{2}\}$ are roots of f mod 3?

3. (a) (5 pts) Let R be a commutative ring of characteristic 2 (recall that this means that $1+1=0$ in R). Prove that for all $a, b \in R$, $(a+b)^2 = a^2 + b^2$. Deduce that more generally, for all $a_1, a_2, \cdots, a_n \in R$,

   $$(a_1 + a_2 + \cdots + a_n)^2 = a_1^2 + a_2^2 + \cdots + a_n^2.$$

   (b) (15 pts) Use the previous question to determine the factorisation of f in $(\mathbb{Z}/2\mathbb{Z})[x]$. Do not forget to prove that your factorisation is complete!

4. (3 pts) Suppose that $g(x)$ is a factor of $f(x)$ in $\mathbb{Z}[x]$. Prove that the leading coefficient of $g(x)$ is ± 1.

5. (25 pts) Prove that $f(x)$ is irreducible in $\mathbb{Z}[x]$.

   Hint: Suppose not. In view of the previous questions, what could be the degrees of the factors of $f(x)$?

6. (10 pts) Prove that $f(x)$ is irreducible in $\mathbb{Q}[x]$.

7. (10 pts) Let $\alpha \in \mathbb{C}$ be one of the roots of $f(x)$, and let $K = \mathbb{Q}(\alpha)$. Prove that $[K : \mathbb{Q}] = 4$, and give a $\mathbb{Q}$-basis of K.

8. (15 pts) Express $\frac{1}{\alpha - 1}$ as a polynomial in α with rational coefficients.

This was the only mandatory exercise, that you must submit before the deadline. The following exercises are not mandatory; they is not worth any points, and you do not have to submit them. However, you can try to solve them for practice, and you are welcome to email me if you have questions about them. The solutions will be made available with the solution to the mandatory exercises.

Exercise 2 *Some irreducible polynomials*

1. Prove that $x^5 + 6x + 12$ is irreducible in $\mathbb{Z}[x]$ and in $\mathbb{Q}[x]$.
2. Let $f = x^7y + y^5 - xy^3 + 2x \in \mathbb{C}[x, y]$. Express f as an element of $\mathbb{C}[y][x]$, and then as an element of $\mathbb{C}[x][y]$. Prove that f is irreducible in $\mathbb{C}[x, y]$.

Exercise 3 *Computations in an extension of* $\mathbb{Q}$

Let $F(x) = x^3 + 2x - 2$, let $\alpha \in \mathbb{C}$ be a root of F, and let $K = \mathbb{Q}(\alpha)$.

1. Prove that $[K : \mathbb{Q}] = 3$.
2. Find $a, b, c \in \mathbb{Q}$ such that $\alpha^4 = a\alpha^2 + b\alpha + c$. Are a, b, c unique?
3. Find $d, e, f \in \mathbb{Q}$ such that $\frac{1}{\alpha^2+\alpha+3} = d\alpha^2 + e\alpha + f$. Are d, e, f unique?
4. Does $\sqrt{2} \in K$?

   Hint: Think in terms of degrees.
5. Find all fields L such that $\mathbb{Q} \subseteq L \subseteq K$.
6. Prove that $\mathbb{Q}(\alpha^2) = K$.