Rings, fields, and modules
Exercise sheet 1

https://www.maths.tcd.ie/~mascotn/teaching/2021/MAU22102/index.html

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Email your answers to aylwarde@tcd.ie by Friday February 19, 4PM.

Exercise 1 *Arithmetic on ideals (100 pts)*

In this exercise, we fix a commutative ring R, and given elements $r_1, \cdots, r_m$ of R, we write $(r_1, \cdots, r_m)$ for the ideal generated by $r_1, \cdots, r_m$; this generalises the notation (r) for principal ideals.

Reminder: In order to prove an if and only if, it is often convenient to prove both implications separately. In order to prove that two subsets A and B are equal, it is often convenient to prove separately that $A \subseteq B$ and that $B \subseteq A$.

1. (10 pts) Let $I \subseteq R$ be an ideal, and let $r \in R$. Prove that $r \in I$ if and only if $(r) \subseteq I$.

2. (a) (10 pts) Let $r_1, \cdots, r_m \in R$. Prove that

$$(r_1, \cdots, r_m) = \{x_1 r_1 + \cdots + x_m r_m \mid x_1, \cdots, x_m \in R\}.$$

   (b) (5 pts) Let $r_1, \cdots, r_m \in R$. Prove that

$$(r_1, \cdots, r_m) = (r_1) + \cdots + (r_m).$$

   (c) (10 pts) Let $r, s \in R$. Prove that $(r)(s) = (rs)$.

   (d) (10 pts) More generally, prove that if $r_1, \cdots, r_m, s_1, \cdots, s_n \in R$, then

$$(r_1, \cdots, r_m)(s_1, \cdots, s_n) = (r_1 s_1, \cdots, r_1 s_n, r_2 s_1, \cdots, r_2 s_n, \cdots, r_m s_1, \cdots, r_m s_n).$$

3. (15 pts) In this question, we take $R = \mathbb{Z}[x]$, and we consider the ideal $I = \{P(x) \in \mathbb{Z}[x] \mid P(0) \text{ is even}\} \subset R$. Prove that $I = (2, x)$.

4. (15 pts) In this question, we take $R = \mathbb{Z}$, and we admit without proof the following facts:

   - *For all $n, m \in \mathbb{Z}$, $\gcd(m, n)$ divides m and n and is of the form $mu + nv$ with $u, v \in \mathbb{Z}$.*
   - *For all $n, m \in \mathbb{Z}$, every common multiple of m and n is a multiple of the lowest common multiple $\mathrm{lcm}(m, n)$.*
   - *Every ideal of $\mathbb{Z}$ is principal.*

Let $m, n \in \mathbb{Z}$. Since every ideal of $\mathbb{Z}$ is principal, there must exist $a, b, c \in \mathbb{Z}$ such that $(m) + (n) = (a)$, $(m) \cap (n) = (b)$, and $(m)(n) = (c)$. Express a, b, c in terms of m and n, and prove that your answer is correct.

5. In this question, R is a general commutative ring, and I and J are ideals of R.

   (a) (15 pts) Prove that
   $$IJ \subseteq I \cap J \subseteq I \subseteq I + J.$$

   (b) (10 pts) Find an example where all these inclusions are strict.
   Hint: Take $R = \mathbb{Z}$, and use the previous question.

This was the only mandatory exercise, that you must submit before the deadline. The following exercises are not mandatory; they are not worth any points, and you do not have to submit them. However, I highly recommend that you try to solve them for practice, and you are welcome to email me if you have questions about them. The solutions will be made available with the solution to the mandatory exercises.

Exercise 2 *Associate elements*

Let R be a commutative **domain**, and let $x, y \in R$. Recall the notation

$$(x) = \{xz \mid z \in R\} \subseteq R,$$

for the ideal generated by x, and similarly for (y).

1. Prove that $(x) \subseteq (y)$ if and only if there exists $z \in R$ such that $x = yz$ (in other words, if $x \in (y)$).
2. Deduce that $(x) = (y)$ if and only if there exists a unit $u \in R^\times$ such that $x = uy$.

Exercise 3 *Products of rings*

Let R_1 an R_2 be two rings, neither of which is the 0 ring. Consider the set of pairs

$$R_1 \times R_2 = \{(x_1, x_2) \mid x_1 \in R_1, x_2 \in R_2\}.$$

1. Let R be a ring, and suppose we have a ring isomorphism
   $$\phi : R_1 \times R_2 \xrightarrow{\sim} R$$
   between a product ring $R_1 \times R_2$ of nonzero rings and R. Prove that there exists an $e \in R$ such that $e^2 = e$ but $e \neq 0$ and $e \neq 1$. Use this to deduce that R cannot be a domain.

   Hint: Take a look at the pair $(1, 0) \in R_1 \times R_2$.

2. Prove that conversely, if there exists $e \in R$ such that $e^2 = e$ but $e \neq 0$ and $e \neq 1$, then R is isomorphic to the product of two nonzero rings.

   Hint: $R_1 = eR$, $R_2 = (1-e)R$.

3. Prove that the ring

$$F = \{f : \mathbb{R} \longrightarrow \mathbb{R} \mid f \text{ continuous}\}$$

   of continuous functions from $\mathbb{R}$ to $\mathbb{R}$, equipped as usual with the laws

$$(f+g)(x) = f(x) + g(x), \quad (fg)(x) = f(x)g(x)$$

   for all $f, g \in F$ and $x \in \mathbb{R}$, is NOT isomorphic to a product ring $R_1 \times R_2$.

   Hint: Proceed by contradiction. You may use without proof the following consequence of the intermediate value theorem: If $f : \mathbb{R} \longrightarrow \mathbb{R}$ is continuous and satisfies $f(x) \in \{0, 1\}$ for all $x \in \mathbb{R}$, then f is constant (and thus either identically 0 or 1).

4. What happens if we drop the continuity condition, and consider instead the ring of all functions from $\mathbb{R}$ to $\mathbb{R}$?