Introduction to number theory
Exercise sheet 6

https://www.maths.tcd.ie/~mascotn/teaching/2020/MAU22301/index.html

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Answers are due for Monday December 14th, 2PM.
The use of electronic calculators and computer algebra software is allowed.

Exercise 1 *Pell-Fermat (100pts)*

1. (25pts) Determine the coefficients a_n of the continued fraction expansion of $\sqrt{14}$.
2. (25pts) Use the previous question to find a fundamental unit in $\mathbb{Z}[\sqrt{14}]$. What is its norm?
3. (20pts) Find integers $x, y \in \mathbb{N}$ such that $x^2 - 14y^2 = 1$ and $y > 100$. How many such pairs (x, y) are there?
4. (30pts) Find integers $x, y \in \mathbb{N}$ (in particular, $y \neq 0$) such that $x^2 - 41y^2 = 1$.

This was the only mandatory exercise, that you must submit before the deadline. The following exercises are not mandatory; they are not worth any points, and you do not have to submit them. However, I highly recommend that you try to solve them for practice, and you are welcome to email me if you have questions about them. The solutions will be made available with the solution to the mandatory exercise.

Exercise 2 *The battle of Hastings*

The battle of Hastings, which took place on October 14, 1066, was a major battle in History.

The following fictional historical text, taken from *Amusement in Mathematics* (H. E. Dundeney, 1917), refers to it:

"The men of Harold stood well together, as their wont was, and formed thirteen squares, with a like number of men in every square thereof. (…) When Harold threw himself into the fray the Saxons were one mighty square of men, shouting the battle cries 'Ut!', 'Olicrosse!', 'Godemite!'."

Use continued fractions to determine how many soldiers this fictional historical text suggests Harold II had at the battle of Hastings.

Exercise 3 *Continued fraction vs. series*

Let $x \in (0,1)$ be irrational, and let $[a_0, a_1, \cdots, a_n] = p_n/q_n$ ($n \in \mathbb{Z}_{\geq 0}$) be the convergents of the continued fraction expansion of x. Prove that

$$x = \sum_{n=0}^{+\infty} \frac{(-1)^n}{q_n q_{n+1}}.$$

Hint: Where could the $(-1)^n$ *come from ?*