Coláiste na Tríonóide, Baile Átha Cliath
Trinity College Dublin
Ollscoil Átha Cliath | The University of Dublin

Faculty of Engineering, Mathematics and Science

School of Mathematics

JS/SS Maths/TP/TJH **Semester 2, 2019**

MAU22102 Rings, fields, and modules — Review exam

Dr. Nicolas Mascot

Instructions to Candidates:

This is a review exam, meant to help you prepare for the actual exam.

Question 1 *Irreducibility*

1. Let K be a field. Determine the units of the polynomial ring $K[x]$. Explain.
2. Let R be a commutative ring. Define what it means for an element of R to be *irreducible*. Spell out the definition in the case $R = K[x]$, where K is a field as above.
3. Let again K be a field. For which non-negative integers $n \geqslant 0$ is the polynomial x^n irreducible in $K[x]$?
4. Give an example of an element of $\mathbb{Q}[x]$ which has degree 2020 and is irreducible.

Question 2 *Radicals and extensions*

Let $\alpha = \sqrt{2}i \in \mathbb{C}$, so that $\alpha^2 = -2$, and let $K = \mathbb{Q}(\alpha)$.

1. Prove that α is algebraic over $\mathbb{Q}$, and determine its minimal polynomial.
2. Determine $[K : \mathbb{Q}]$, and find a $\mathbb{Q}$-basis of K.
3. Let $\beta = \sqrt{2}$. Using the previous question, prove that $\beta \notin K$.
4. Is it possible to prove that $\beta \notin K$ by degree considerations only?
5. Determine the minimal polynomial of α over K. Comment
6. Prove that β is algebraic over K, and determine its minimal polynomial over K. Also
7. Let $L = K(\beta)$. Determine $[L : \mathbb{Q}]$, and find a $\mathbb{Q}$-basis of L.
8. Prove that $i \in L$. What are its coordinates on the $\mathbb{Q}$-basis of L that you fond at the previous question?
9. Is it true that $L = \mathbb{C}$?

Question 3 *Annihilators and torsion elements*

Let R be a commutative domain, and let M be an R-module. Given an element $m \in M$, we define its *annihilator* as the subset

$$\mathrm{Ann}(m) = \{r \in R \mid rm = 0\}$$

of R.

1. An example: determine $\mathrm{Ann}(m)$ if $m = 0$.

2. Prove that for any $m \in M$, $\mathrm{Ann}(m)$ is an ideal of R.

3. We say that an element $m \in M$ is *torsion* if its annihilator is not reduced to $\{0\}$, i.e. if there exists $r \in R$, $r \neq 0$ such that $rm = 0$, and we define

$$M_{\mathrm{tor}} = \{m \in M \mid m \text{ is torsion}\}.$$

   Prove that M_{tor} is a submodule of M.

4. We say that M is *torsion-free* if $M_{\mathrm{tor}} = \{0\}$. Prove that if M is free of finite rank, then M is torsion-free.

5. Prove that for any module M, the quotient module M/M_{tor} is torsion-free.