Fields, rings, and modules
Exercise sheet 3

https://www.maths.tcd.ie/~mascotn/teaching/2020/MAU22102/index.html

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Answers are due for Thursday March 19, 4PM.

Exercise 1 *Computations in an extension of $\mathbb{Q}$ (100 pts)*

Let $F(x) = x^3 + 2x - 2$, let $\alpha \in \mathbb{C}$ be a root of F, and let $K = \mathbb{Q}(\alpha)$.

1. (15 pts) Prove that $[K : \mathbb{Q}] = 3$.
2. (10 pts) Find $a, b, c \in \mathbb{Q}$ such that $\alpha^4 = a\alpha^2 + b\alpha + c$. Are a, b, c unique?
3. (15 pts) Find $d, e, f \in \mathbb{Q}$ such that $\frac{1}{\alpha^2+\alpha+3} = d\alpha^2 + e\alpha + f$. Are d, e, f unique?
4. (20 pts) Does $\sqrt{2} \in K$?
   Hint: Think in terms of degrees.
5. (20 pts) Find all fields L such that $\mathbb{Q} \subseteq L \subseteq K$.
6. (20 pts) Prove that $\mathbb{Q}(\alpha^2) = K$.