Fields, rings, and modules
Exercise sheet 2

https://www.maths.tcd.ie/~mascotn/teaching/2020/MAU22102/index.html

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Answers are due for Thursday March 12th, 4PM.

Exercise 1 *Factorisation of polynomials (100 pts)*

Justify your answers carefully in order to receive full credit.

1. (8 pts) Let K be a field, and let $F(x) \in K[x]$ of degree 1. Prove that $F(x)$ is irreducible in $K[x]$.

   Hint: What could be the degrees of the factors?

2. (7 pts) Give an example of a ring R and of a polynomial $F(x) \in R[x]$ of degree 1 such that $F(x)$ is **not** irreducible in $R[x]$.

3. (15 pts) Let K be a field, and let $F(x) \in K[x]$ of degree 2 or 3. Prove that $F(x)$ is reducible in $K[x]$ if and only if it has a root in K.

   Hint: What could be the degrees of the factors?

4. (10 pts) Give an example of a polynomial in $\mathbb{R}[x]$ which is reducible in $\mathbb{R}[x]$ but has no root in $\mathbb{R}$.

5. (15 pts) Let $F(x) \in \mathbb{Z}[x]$ be non-constant and monic, and let $n \geqslant 2$ be an integer. Prove that if $F(x)$ is irreducible in $(\mathbb{Z}/n\mathbb{Z})[x]$, then $F(x)$ is irreducible in $\mathbb{Z}[x]$.

   Hint: Proceed by contradiction.

6. (15 pts) Prove that $F(x) = x^3 + x + 1$ in irreducible in $\mathbb{Z}[x]$ and in $\mathbb{Q}[x]$.

   Hint: Reduce mod 2 and use the previous questions.

7. (10 pts) Prove that $F(x) = x^5 + 4x + 2$ in irreducible in $\mathbb{Z}[x]$ **and** in $\mathbb{Q}[x]$.

8. (20 pts) Prove that $F(x, y) = x^5 - x^3y + xy^7 + 2y$ is irreducible in $\mathbb{C}[x, y]$.