Fields, rings, and modules
Exercise sheet 1

https://www.maths.tcd.ie/~mascotn/teaching/2020/MAU22102/index.html

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Answers are due for Thursday February 13rd, 4PM.

Exercise 1 *Associate elements (40 pts)*

Let R be a commutative **domain**, and let $x, y \in R$. Recall the notation

$$(x) = \{xz \mid z \in R\} \subseteq R,$$

for the ideal generated by x, and similarly for (y).

1. (20 pts) Prove that $(x) \subseteq (y)$ if and only if there exists $z \in R$ such that $x = yz$ (in other words, if $x \in (y)$).

2. (20 pts) Deduce that $(x) = (y)$ if and only if there exists a unit $u \in R^\times$ such that $x = uy$.

Exercise 2 *Products of rings (60 pts)*

Let R_1 an R_2 be two rings, neither of which is the 0 ring. Consider the set of pairs

$$R_1 \times R_2 = \{(x_1, x_2) \mid x_1 \in R_1, x_2 \in R_2\}.$$

1. (20 pts) Show that the operations

   $$(x_1, x_2) + (y_1, y_2) = (x_1 + y_1, x_2 + y_2), \quad (x_1, x_2) \times (y_1, y_2) = (x_1 \times y_1, x_2 \times y_2)$$

   for all $x_1, y_1 \in R_1$ an $x_2, y_2 \in R_2$ define a ring structure on $R_1 \times R_2$. What are the 0 and the 1 of $R_1 \times R_2$?

   We call $R_1 \times R_2$ equipped with the above operations the product ring *of R_1 and R_2.*

2. (20 pts) Let R be another ring, and suppose we have a ring isomorphism

   $$\phi : R_1 \times R_2 \xrightarrow{\sim} R$$

   between a product ring $R_1 \times R_2$ and R. Prove that there exists an $e \in R$ such that $e^2 = e$ but $e \neq 0$ and $e \neq 1$. Deduce that R cannot be a domain.

   Hint: Take a look at the pair $(1, 0) \in R_1 \times R_2$.

3. (20 pts) Using the previous question, prove that the ring

$$F = \{f : \mathbb{R} \longrightarrow \mathbb{R} \mid f \text{ continuous}\}$$

of continuous functions from $\mathbb{R}$ to $\mathbb{R}$, equipped as usual with the laws

$$(f+g)(x) = f(x) + g(x), \quad (fg)(x) = f(x)g(x)$$

for all $f, g \in F$ and $x \in \mathbb{R}$, is NOT isomorphic to a product ring $R_1 \times R_2$.

Hint: Proceed by contradiction. You may use without proof the following consequence of the intermediate value theorem: If $f : \mathbb{R} \longrightarrow \mathbb{R}$ is continuous and satisfies $f(x) \in \{0, 1\}$ for all $x \in \mathbb{R}$, then f is constant (and thus either identically 0 or 1).