Coláiste na Tríonóide, Baile Átha Cliath
Trinity College Dublin
Ollscoil Átha Cliath | The University of Dublin

Faculty of Engineering, Mathematics and Science

School of Mathematics

JS/SS Maths/TP/TJH **Semester 1, 2019**

MAU34101 Galois theory — Mock exam

Some date **Some location** **Some time**

Dr. Nicolas Mascot

Instructions to Candidates:

This is a mock exam, and is intended for revision purposes only.
This paper contains **five** questions. You **must** attempt **four** of them: question 1, and **exactly three** of questions 2, 3, 4, and 5.
Should you attempt all questions (not recommended), you will **only** get the marks for question 1 and the best three others.
Non-programmable calculators are permitted for this examination.

You may not start this examination until you are instructed to do so by the Invigilator.

Question 1 *Bookwork*

Let $K \subset L$ be a finite extension, and let $\Omega \supset K$ be algebraically closed. Which inequalities do we always have between $[L : K]$, $\# \operatorname{Aut}_K(L)$, $\# \operatorname{Hom}_K(L, \Omega)$? When are they equalities? State equivalent conditions.

Solution 1

We always have

$$\# \operatorname{Aut}_K(L) \leq \# \operatorname{Hom}_K(L, \Omega) \leq [L : K].$$

The left inequality is an equality iff. L is *normal* over K, which means tht there exists $F(x) \in K[x]$ such that L is (K-isomorphic to) the splitting field of F over K. An equivalent characterisation is that any *irreducible* $P(x) \in K[x]$ having one root in L must split completely over L.

The right inequality is an equality iff. L is a separable extension of K, which means that the minpoly over K of any element of L is separable.

Question 2 *Correspondence in degree 3*

Let K be a field, and $F(x) \in K[x]$ be separable and of degree 3. Denote its 3 roots in its splitting field L by $\alpha_1,\ \alpha_2,\ \alpha_3$.

1. What are the possibilities for $\operatorname{Gal}_K(F)$? How can you tell them apart?

2. For each of the cases found in the previous question, sketch the diagram showing all the fields $K \subset E \subset L$ and identifying these fields. In particular, locate $K(\alpha_1)$, $K(\alpha_2)$, $K(\alpha_3)$, $K(\alpha_1, \alpha_2)$, etc.

3. In which of the cases above is the stem field of F isomorphic to its splitting field? (*Warning: there is a catch in this question.*)

Solution 2

Some general remarks first. In any case, $\mathrm{Gal}_K(F)$ is a subgroup of S_3 acting on the roots of F; the only such subgroups are S_3, A_3, $\{\mathrm{Id} \times S_2\}$, and $\{\mathrm{Id}\}$. Besides, we know that $\alpha_1 + \alpha_2 + \alpha_3 \in K$ by Vieta's formulas (it is the negative of the coefficient of x^2 in F), so $\alpha_3 = (\alpha_1 + \alpha_2 + \alpha_3) - \alpha_1 - \alpha_2 \in K(\alpha_1, \alpha_2$; as a result, we always have

$$K(\alpha_1, \alpha_2) = K(\alpha_1, \alpha_2, \alpha_3).$$

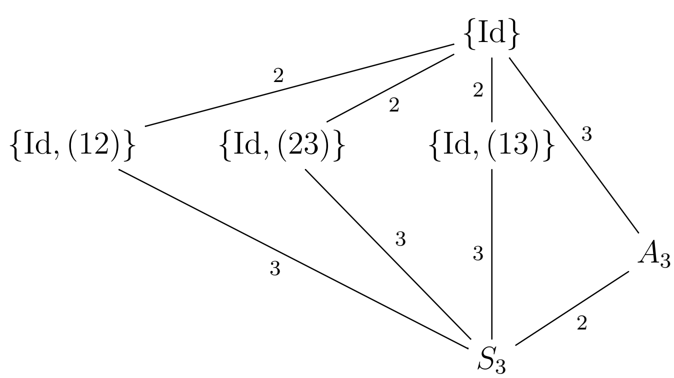
We can also recover this fact by Galois theory: if $\sigma \in \mathrm{Gal}(K(\alpha_1, \alpha_2, \alpha_3)/K(\alpha_1, \alpha_2))$, then $\sigma \in \S_3$ fixes 1 and 2, so it must be the identity. Therefore $K(\alpha_1, \alpha_2, \alpha_3)$ and $K(\alpha_1, \alpha_2)$ both correspond to the same subgroup, namely $\{\mathrm{Id}\}$, so they are the same field.

Similarly, we have

$$K(\alpha_1, \alpha_3) = K(\alpha_2, \alpha_3) = K(\alpha_1, \alpha_2, \alpha_3).$$

Let us now examine the possible cases.

Suppose first that F is irreducible over K, and that $\operatorname{disc} F$ is not a square in K. Then $\mathrm{Gal}_K(F)$ is a transitive subgroup not contained in A_3, so it is S_3. To find the intermediate fields, we start with the subgroups:



Since $\{\mathrm{Id}, (23)\}$ is the stabiliser of α_1, the corresponding field is $K(\alpha_1)$, which is indeed an extension of K of degree 3 since F, being irreducible, is the minpoly of α_1. Similarly for $K(\alpha_2)$ and $K(\alpha_3)$. Finally, let E correspond to A_3; then the extension $E \subset K(\alpha_1, \alpha_2, \alpha_3)$ is

Galois of Galois group A_3, so $\operatorname{disc} F$ is a square in E. Besides $[E : K] = [S_3 : A_3] = 2$ and $\sqrt{\operatorname{disc} F} \notin K$ by assumption, so $E = K(\sqrt{\operatorname{disc} F})$. We thus get

$$\begin{tikzcd}
 & & K(\alpha_1,\alpha_2,\alpha_3) \arrow[dll, dash, "2"'] \arrow[dl, dash, "2"] \arrow[d, dash, "2"] \arrow[ddr, dash, "3"] & \\
K(\alpha_3) \arrow[ddrr, dash, "3"'] & K(\alpha_2) \arrow[ddr, dash, "3"] & K(\alpha_1) \arrow[dd, dash, "3"] & \\
 & & & K(\sqrt{\operatorname{disc} F}) \arrow[dl, dash, "2"] \\
 & & K &
\end{tikzcd}$$

In particular, the stem fields $K(\alpha_1)$, $K(\alpha_2)$, $K(\alpha_3)$, which are all isomorphic (to $K[x]/F(x)$, that's a theorem) but distinct, are smaller than the splitting field $K(\alpha_1, \alpha_2, \alpha_3)$ in this case.

Suppose now that F is irreducible and $\operatorname{disc} F$ is a square in K. Then $\operatorname{Gal}_K(F) = A_3$ since it is transitive and contained in A_3. Since $A_3 \simeq \mathbb{Z}/3\mathbb{Z}$ has prime order, it cannot have any nontrivial subgroup, so by the Galois correspondence the only intermediate fields are

$$\begin{tikzcd}
K(\alpha_1,\alpha_2,\alpha_3) \arrow[d, dash, "\#A_3=3"] \\
K.
\end{tikzcd}$$

Since F is irreducible over K, it has no root in K, so $\alpha_1 \notin K$, so $K(\alpha_1) \supsetneq K$, so

$$K(\alpha_1) = K(\alpha_1, \alpha_2, \alpha_3).$$

We can also see this by noting that the corresponding subgroup is the stabiliser of 1 in A_3, which is reduced to $\{\mathrm{Id}\}$. Similarly

$$K(\alpha_2) = K(\alpha_3) = K(\alpha_1, \alpha_2, \alpha_3).$$

So this time, the stem fields $K(\alpha_1)$, $K(\alpha_2)$, $K(\alpha_3)$ are all the same (not only up to isomorphism), and agree with the splitting field $K(\alpha_1, \alpha_2, \alpha_3)$.

Suppose now that F factors as $1 + 2$ over K, and let α_1 be the root of F in K. Then $F(x) = (x - \alpha_1)G(x)$, where $G(x) = (x - \alpha_2)(x - \alpha_3)$ is irreducible over K. In particular

$\mathrm{Gal}_K(F) = \mathrm{Id} \times \mathrm{Gal}_K(G) = \mathrm{Id} \times S_2$. Again this does not have any nontrivial subgroups, so the only intermediate fields are

$$\begin{array}{c} K(\alpha_1, \alpha_2, \alpha_3) \\ \Big| {\scriptstyle 2} \\ K. \end{array}$$

We have $K(\alpha_1) = K$, but $K(\alpha_2) = K(\alpha_3) = K(\alpha_1, \alpha_2, \alpha_3)$.

Finally, if F factors completely over K, then all the α_i are in K, so the only intermediate field is

$$K = K(\alpha_1, \alpha_2, \alpha_3)$$

which is of course also $K(\alpha_i)$ for any i. This checks out with Galois theory, since in this case $\mathrm{Gal}_K(F) = \{\mathrm{Id}\}$ has only one subgroup (including itself and$\{\mathrm{Id}\}$, which is the same thing in this case).

In the last two cases, there is no stem field anymore since F is not irreducible (that was the catch).

Question 3 *Galois group computations*

Determine the Galois group over $\mathbb{Q}$ of the polynomials below, and say if they are solvable by radicals over $\mathbb{Q}$.

1. $x^3 - x^2 - x - 2$,
2. $x^3 - 3x - 1$,
3. $x^3 - 7$,
4. $x^5 + 21x^2 + 35x + 420$,
5. $x^{10} + x^9 + x^8 + x^7 + x^6 + x^5 + x^4 + x^3 + x^2 + x + 1$.

Solution 3

1. Looking for rational roots, we find the factorisation $f = (x-2)(x^2+x+1)$. The second factor has $\Delta = -3 < 0$ so is irreducible over $\mathbb{R}$ and hence over $\mathbb{Q}$. As a result, the polynomial is separable and has Galois group $\{\mathrm{Id}\} \times S_2$. This is Abelian, hence solvable, so this polynomial is solvable by radicals.

2. No rational roots, so irreducible (since degree 3). $\mathrm{disc} = 81 = 9^2$ so A_3. This group is Abelian, hence solvable, so this polynomial is solvable by radicals.

3. No rational roots, so irreducible (since degree 3). $\mathrm{disc} = -3^3 \cdot 7^2$ is clearly not a square in $\mathbb{Q}$, so S_3. This group is solvable because $\mathrm{Id} \triangleleft A_3 \triangleleft S_3$ has Abelian factors, so this polynomial is solvable by radicals.

   Note: since S_3 is solvable, any subgroup is also solvable, so any equation of degree 3 is solvable by radicals.

4. Eisenstein at 7 so irreducible, so transitive Galois group. Mod 2, factors as

   $$x^5 + x^2 + x = x(x^4 + x + 1).$$

   The second factor is irreducible: if not, it would have a factor of degree 1 or 2, but

   $$\gcd(x^4+x+1, x^{2^2}-1) = \gcd(x^4+x+1, x^4-1-(x^4+x+1)) = \gcd(x^4+x+1, x) = 1$$

   so it has no irreducible factor of degree dividing 2. So we have a 4-cycle.

   Mod 3, factors as

   $$x^5 - x = (x-1)x(x+1)(x^2+1)$$

   with x^2+1 irreducible mod 3 (degree ≤ 3, no roots), so we have a 2-cycle. Conclusion : S_5. We know that this is not a solvable group, so this polynomial is not solvable by radicals.

5. This is the cyclotomic polynomial $\Phi_{11}(x)$, so Galois group $(\mathbb{Z}/11\mathbb{Z})^\times$. This is Abelian, hence solvable, so this polynomial is solvable by radicals even though it hs degree ≥ 5 (indeed, the roots are $\sqrt[11]{1}$...)

Question 4 *A cosine formula (35 pts)*

Let $c = \cos(2\pi/17)$.

1. Prove that c is algebraic over $\mathbb{Q}$.
2. Determine the conjugates of c over $\mathbb{Q}$, and its degree as an algebraic number over $\mathbb{Q}$.
3. Explain how one could in principle use Galois theory (and a calculator / computer) to find an explicit formula for c.

Solution 4

1. Let $\zeta = exp(2\pi i/17)$, a primitive 17-th root of 1. Since ζ is clearly algebraic over $\mathbb{Q}$ (as a root of $x^{17} - 1$ / even better: of $\Phi_{17}(x)$), $\mathbb{Q}(\zeta)$ is a finite extension of $\mathbb{Q}$. As a result, it is an algebraic extension of $\mathbb{Q}$, which means that all its elements are algebraic over $\mathbb{Q}$. This applies in particular to $c = \frac{\zeta+\zeta^{-1}}{2}$.

2. Let ζ as above, and $L = \mathbb{Q}(\zeta)$. We know that L is Galois over $\mathbb{Q}$; since $c \in L$, this implies that the conjugates of c are the $\sigma(c)$ for $\sigma \in \mathrm{Gal}(L/\mathbb{Q})$. It remains to determine them explicitly.

   First, we know that $\mathrm{Gal}(L/\mathbb{Q}) \simeq (\mathbb{Z}/17\mathbb{Z})^\times$. Second, this group is cyclic (of order 16 of course) because 17 is prime. Let us look for a generator. 2 does not work because $2^4 = 16 \equiv -1 \bmod 17$, so $2^8 = 1$, so 2 has order $8 < 16$. However 3 is a generator since

   $$3^2 = 9,\ 3^4 = 9^2 = 81 \equiv -4,\ 3^8 \equiv (-4)^2 \equiv -1.$$

   As a result, $(\mathbb{Z}/17\mathbb{Z})^* \simeq \mathbb{Z}/16\mathbb{Z} = \langle 3 \rangle$, so $\mathrm{Gal}(L/\mathbb{Q})$ is generated by $\sigma_3 : \zeta \mapsto \zeta^3$.

   In particular, the conjugates of c are its orbit under σ_3. Using $c = \frac{\zeta+\zeta^{-1}}{2}$ (and some patience), we compute that

   $$\sigma_3(c) = \frac{\zeta^3+\zeta^{-3}}{2} = \cos(6\pi/17),$$

   $$\sigma_3^2(c) = \frac{\zeta^9+\zeta^{-9}}{2} = \cos(18\pi/17) = \frac{\zeta^{-8}+\zeta^8}{2} = \cos(19\pi/17),$$

$$\sigma_3^3(c) = \frac{\zeta^{27}+\zeta^{-27}}{2} = \frac{\zeta^{-7}+\zeta^{7}}{2} = \cos(14\pi/17),$$
$$\sigma_3^4(c) = \frac{\zeta^{-21}+\zeta^{21}}{2} = \frac{\zeta^{-4}+\zeta^{4}}{2} = \cos(8\pi/17),$$
$$\sigma_3^5(c) = \frac{\zeta^{-12}+\zeta^{12}}{2} = \frac{\zeta^{5}+\zeta^{-5}}{2} = \cos(10\pi/17),$$
$$\sigma_3^6(c) = \frac{\zeta^{15}+\zeta^{-15}}{2} = \frac{\zeta^{-2}+\zeta^{2}}{2} = \cos(4\pi/17),$$
$$\sigma_3^7(c) = \frac{\zeta^{-6}+\zeta^{6}}{2} = \cos(12\pi/17),$$
$$\sigma_3^8(c) = \frac{\zeta^{-18}+\zeta^{18}}{2} = \frac{\zeta+\zeta^{-1}}{2} = \cos(2\pi/17) = c,$$

so we stop here (note that since $3^8 \equiv -1$, we already knew that σ_3^8 would fix c, so the orbit would have length $\leqslant 8$): the conjugates of c are

$$c = \cos(2\pi/17),\ \cos(6\pi/17),\ \cos(18\pi/17),\ \cos(14\pi/17),$$

$$\cos(8\pi/17),\ \cos(10\pi/17),\ \cos(4\pi/17),\ \cos(12\pi/17).$$

Using a calculator, one checks that they are all distinct. Since they are the roots of the minimal polynomial of c, we see that the degree of c as an algebraic number is 8.

3. Since $\mathrm{Gal}(L/\mathbb{Q})$ is cyclic of order 16, it has precisely one subgroup of the each of the following orders: 1, 2, 4, 8, 16 (and these all all its subgroups). The Galois correspondence show that there is a succession of extensions of degree 2 starting at $\mathbb{Q}$ an culminating at L. This are all the subfields of L (since these were all the subgroups). The field $\mathbb{Q}(c)$ must be one of them; since this field has degree 8 over $\mathbb{Q}$ by the above, it is actually the second-to-top one (the top one being L).

   Starting with $\mathbb{Q}$, we can now find an explicit generator by expressing a generator in terms of ζ, finding its other conjugate over the subfield just below it by using the Galois action (there will be only one other conjugate since each extension step is of degree 2), deducing its minimal polynomial over that subfield, and solving it (which we can since it will have degree 2).

   For instance, for the first step, we see that $\alpha = \sum_{k=0}^{7} \sigma_3^{2k}(\zeta)$ lies in the extension of degree 2 over Q since it is fixed by σ_3^2 (which generates the corresponding subgroup of order 8), and has $\alpha' = \sigma_3(\alpha) = \sum_{k=0}^{7} \sigma_3^{2k+1}(\zeta)$ as a conjugate. Sine one checks with a calculator that $\alpha' \neq \alpha$, we have that α generates the extension of degree 2 (and so does α'), and satisfies its minimal polynomial $A(x) = (x-\alpha)(x-\alpha') \in \mathbb{Q}[x]$. Expressing it in terms of ζ (which is really painful without a computer) yields $A(x) = x^2 + x - 4$, which shows that $\alpha, \alpha' = \frac{-1 \pm \sqrt{17}}{2}$, so this extension is actually $\mathbb{Q}(\sqrt{17})$.

Next, we find similarly that $\beta = \sum_{k=0}^{3} \sigma_3^{4k}(\zeta)$ lies in the extension of degree 4, and generates it since it is distinct from its conjugate $\beta' = \sigma_3(\beta)$ over $\mathbb{Q}(\alpha)$; and since it is a root of $B(x) = (x-\beta)(x-\beta')$ which must lie in $\mathbb{Q}(\alpha)[x]$, we can express it in terms of α.

With a lot of courage (or in my case, a good computer program), we find that $B(x) = x^2 - \alpha + 1$ whence $\beta, \beta' = \frac{\alpha \pm \sqrt{\alpha^2 - 4}}{2}$. Continuing this way, we finally arrive to the fantastically horrible formula

$$\cos \frac{2\pi}{17} = \frac{-1 + \sqrt{17} + \sqrt{2}\sqrt{17 - \sqrt{17}} + 2\sqrt{17 + 3\sqrt{17} - \sqrt{170 + 38\sqrt{17}}}}{16}.$$

Question 5 *Extensions of finite field are Galois (35 pts)*

Let $p \in \mathbb{N}$ be prime, $n \in \mathbb{N}$, and $q = p^n$.

1. Give two proofs of the fact that the extension $\mathbb{F}_p \subset \mathbb{F}_q$ is Galois: one by viewing $\mathbb{F}_q$ as a splitting field, and the other by considering the order of $\mathrm{Frob} \in \mathrm{Aut}(\mathbb{F}_q)$.

2. What does the Galois correspondence tell us for $\mathbb{F}_p \subset \mathbb{F}_q$?

3. Generalise to an arbitrary extension of finite fields $\mathbb{F}_q \subset \mathbb{F}_{q'}$.

Solution 5

1. Recall that

$$\mathbb{F}_q = \{x \in \overline{\mathbb{F}_p} \mid x^q = x\}.$$

In particular, $\mathbb{F}_q$ is the splitting field over $\mathbb{F}_p$ of $F(x) = x^q - x$, so it is normal over $\mathbb{F}_p$; besides, $F' = -1$ has no common factor with F, so F is separable, so $\mathbb{F}_q$ is separable over $\mathbb{F}_p$ (we may also argue that $\mathbb{F}_p$, being finite, is perfect).

Second proof: $\mathrm{Frob} : x \mapsto x^p \in \mathrm{Aut}_{\mathbb{F}_p}(\mathbb{F}_q)$. Its iterates are $\mathrm{Frob}^k : x \mapsto x^{p^k}$, so if Frob has order o, then every element of $\mathbb{F}_q$ is a root of $x^{p^o} - x$, whence $p^o \geqslant q$ by considering the degree, i.e. $o \geqslant n$. SO Frob has at least n distinct iterates in $\mathrm{Aut}_{\mathbb{F}_p}(\mathbb{F}_q)$, so the inequality

$$\# \mathrm{Aut}_{\mathbb{F}_p}(\mathbb{F}_q) \leq [\mathbb{F}_q : \mathbb{F}_p] = n$$

is an equality, so the extension is Galois (cf. question 1). Besides, this proof also show that the Galois group is cyclic and generated by Frob.

2. The subgroups of

$$\mathrm{Gal}(\mathbb{F}_q/\mathbb{F}_p) = \langle \mathrm{Frob} \rangle \simeq \mathbb{Z}/n\mathbb{Z}$$

are the

$$\langle \mathrm{Frob}^d \rangle \simeq d\mathbb{Z}/n\mathbb{Z}$$

for $d \mid n$ since the former is cyclic by the above. For each d, the corresponding subfield is

$$\mathbb{F}_q^{\langle \mathrm{Frob}^d \rangle} = \{x \in \mathbb{F}_q \mid x^{p^d} = x\} = \mathbb{F}_{p^d}$$

as predicted by the classification of finite fields.

3. By the same arguments as the above, this extension is Galois, with cyclic Galois group generated by $\mathrm{Frob}_q : x \mapsto x^q$ (since it must induce the identity on $\mathbb{F}_q$). The Galois correspondence then shows that the intermediate fields are the $\mathbb{F}_{q^d}$ for $d \mid m$, where $q' = q^m$, as predicted by the classification of finite fields.