Coláiste na Tríonóide, Baile Átha Cliath
Trinity College Dublin
Ollscoil Átha Cliath | The University of Dublin

Faculty of Engineering, Mathematics and Science

School of Mathematics

JS/SS Maths/TP/TJH **Semester 1, 2019**

MAU34101 Galois theory

Wednesday December 11 RDS Simmonscourt 17:00 – 19:00

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Instructions to Candidates:

This exam paper contains **four** questions. You **must** attempt **three** of them: question 1, and **exactly two** of questions 2, 3, and 4.
Should you attempt all questions (not recommended), you will **only** get the marks for question 1 and the best two others.
Non-programmable calculators are permitted for this examination.

You may not start this examination until you are instructed to do so by the Invigilator.

Question 1 *Bookwork (30 pts)*

1. *(10 pts)* Let K be a field, and $P(x) \in K[x]$ be an irreducible polynomial. Give the definition of the *stem field* and of the *splitting field* of $P(x)$ over K. Give an example of K and $P(x)$ where the stem field and the splitting field are not the same.

2. *(10 pts)* State the Galois correspondence.

3. *(10 pts)* Let K be a field, let $F(x) \in K[x]$, and let G be the Galois group of $F(x)$ over K. Which property must G have for $F(x)$ to be solvable by radicals over K? Explain what this property means in terms of subgroups of G. Give an example of a group G that satisfies this property, and of one that does not (no justification needed).

Question 2 *Nested radicals (35 pts)*

Let $\alpha = \sqrt{3+\sqrt{5}}$ and $\beta = \sqrt{3-\sqrt{5}}$, so that α and β are both roots of $F(x) = (x^2-3)^2-5$. Finally, let $L = \mathbb{Q}(\alpha)$.

1. *(6 pts)* Prove that $[L : \mathbb{Q}] = 4$.

2. *(7 pts)* Prove that L is a Galois extension of $\mathbb{Q}$.

   Hint: Compute $\alpha\beta$.

3. *(6 pts)* Prove that $\mathrm{Gal}(L/\mathbb{Q}) \simeq (\mathbb{Z}/2\mathbb{Z}) \times (\mathbb{Z}/2\mathbb{Z})$.

4. *(8 pts)* Sketch a diagram showing all fields E such that $\mathbb{Q} \subseteq E \subseteq L$, and identifying these fields explicitly. Justify your answer.

   Hint: Compute $(\alpha+\beta)^2$.

5. *(8 pts)* Prove that $F(x)$ is reducible mod p for every prime number $p \in \mathbb{N}$.

Question 3 *A polynomial with Galois group A_4 (35 pts)*

Let $F(x) = x^4 - 2x^3 + 2x^2 + 2 \in \mathbb{Q}[x]$. We denote the roots of $F(x)$ in $\mathbb{C}$ by α_1, α_2, α_3, and α_4.

In this exercise, you may use without proof the following facts:

- *The discriminant of f is $\Delta_f = 3136 = 2^6 \cdot 7^2$.*
- *The transitive subgroups of the symmetric group S_4 are*
  - *S_4 itself,*
  - *the alternate group A_4,*
  - *the dihedral group D_8 of symmetries of the square,*
  - *the Klein group $V_4 = \{\mathrm{Id}, (12)(34), (13)(24), (14)(23)\} \simeq (\mathbb{Z}/2\mathbb{Z}) \times (\mathbb{Z}/2\mathbb{Z})$,*
  - *and the cyclic group $\mathbb{Z}/4\mathbb{Z}$.*

1. *(2 pts)* Show that $F(x)$ is irreducible over $\mathbb{Q}$.
2. *(7 pts)* Show that $F(x)$ factors mod 3 as a linear factor times an irreducible factor of degree 3.
3. *(8 pts)* Show that the Galois group of $F(x)$ is A_4.
4. *(9 pts)* Prove that $\mathbb{Q}(\alpha_1, \alpha_2, \alpha_3, \alpha_4) = \mathbb{Q}(\alpha_1, \alpha_2)$.
5. *(9 pts)* Determine the degrees of the irreducible factors of $F(x)$ over $\mathbb{Q}(\alpha_1)$.

Question 4 *Abelian Galois group (35 pts)*

Let K be a field, $P(x) \in K[x]$ irreducible and separable, $L \supset K$ the splitting field of $P(x)$ over K, and $\alpha \in L$ a root of $P(x)$.

1. *(5 pts)* Explain why L is a Galois extension of K.
2. *(30 pts)* We now suppose that the group $\mathrm{Gal}(L/K)$ is Abelian. Prove that $K(\alpha) = L$.

   Hint: What does the fact that $\mathrm{Gal}(L/K)$ is Abelian imply about its subgroups?