Galois theory — Exercise sheet 2

https://www.maths.tcd.ie/~mascotn/teaching/2019/MAU34101/index.html

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Answers are due for Tuesday October 29th, 3PM.

Exercise 1 *Yes or no (35 pts)*

Let $f(x) = x^3 + x + 1 \in \mathbb{Q}[x]$ (you may assume without proof that f is irreducible over $\mathbb{Q}$), and let $L = \mathbb{Q}[x]/(f)$.

1. (10 pts) Is L a separable extension of $\mathbb{Q}$? Explain.
2. (20 pts) Is L a normal extension of $\mathbb{Q}$? Explain.

   Hint: What does the fact that $f : \mathbb{R} \longrightarrow \mathbb{R}$ is strictly increasing tell you about the complex roots of f?
3. (5 pts) Is L a Galois extension of $\mathbb{Q}$? Explain.

Exercise 2 *Square roots (65 pts)*

Let $L = \mathbb{Q}(\sqrt{10}, \sqrt{42})$.

In this exercise, you may use without proof the fact that for all $a, b \in \mathbb{Q}^\times$,

$$\begin{aligned}\mathbb{Q}(\sqrt{a}) = \mathbb{Q}(\sqrt{b}) &\iff \sqrt{b} \in \mathbb{Q}(\sqrt{a}) \\ &\iff a/b \text{ is a square in } \mathbb{Q} \\ &\iff \text{The numerator and denominator of } a/b \text{ are squares.}\end{aligned}$$

1. (5 pts) Prove that L is a Galois extension of $\mathbb{Q}$.
2. (10 pts) Prove that $[L : \mathbb{Q}] = 4$.
3. (15 pts) Describe all the elements of $\mathrm{Gal}(L/\mathbb{Q})$. What is $\mathrm{Gal}(L/\mathbb{Q})$ isomorphic to?
4. (20 pts) Sketch the diagram showing all intermediate extensions $\mathbb{Q} \subset E \subset L$, ordered by inclusion (you may re-use without proof the subgroup diagram of $\mathrm{Gal}(L/\mathbb{Q})$ seen in class). Explain clearly which field corresponds to which subgroup.
5. (15 pts) Does $\sqrt{15} \in L$? Use the previous question to answer.