Math 261 — Exercise sheet 9

http://staff.aub.edu.lb/~nm116/teaching/2018/math261/index.html

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The use of calculators is recommended, at least for the first exercise.

Exceptionally, none of these exercises are mandatory.

Exercise 9.1: Continued fraction expansions

1. Express the rational $\dfrac{2018}{261}$ as a continued fraction.

2. Compute the continued fraction expansion of $\dfrac{e+1}{e-1}$ where

$$e = \exp(1) \approx 2.718281828.$$

   Can you see a pattern? Also compute the 4 first convergents, and see how many correct decimals they have compared to $\frac{e+1}{e-1}$.

3. Assuming that the pattern you spotted in the previous question does hold, prove that e is irrational.

Exercise 9.2: A Pell-Fermat equation

1. Compute the continued fraction expansion of $\sqrt{6}$.

2. Use the previous question to find the fundamental solution to the equation $x^2 - 6y^2 = 1$.

3. Use the ring structure of $\mathbb{Z}[\sqrt{6}]$ to find 2 other non-trivial solutions (changing the signs of x and y does not count !)

Exercise 9.3: The battle of Hastings

The battle of Hastings was a major battle in English history. It took place on October 14, 1066.

The following fictional historical text, taken from *Amusement in Mathematics* (H. E. Dundeney, 1917), refers to it:

"The men of Harold stood well together, as their wont was, and formed thirteen squares, with a like number of men in every square thereof. (...) When Harold threw himself into the fray the Saxons were one mighty square of men, shouting the battle cries 'Ut!', 'Olicrosse!', 'Godemite!'."

Use continued fractions to determine how many soldiers this fictional historical text suggests Harold II had at the battle of Hastings.

Exercise 9.4: More Pell-Fermat

Redo exercise 9.2, with 6 replaced by 14, 15, 17, and 18.

Exercise 9.5: Continued fraction vs. series

Let $x \in (0,1)$ be irrational, and let $[a_0, a_1, \cdots, a_n] = p_n/q_n$ ($n \in \mathbb{N}$) be the convergents of the continued fraction expansion of x. Prove that

$$x = \sum_{n=0}^{+\infty} \frac{(-1)^n}{q_n q_{n+1}}.$$

Hint: Where could the $(-1)^n$ *come from ?*