Math 261 — Exercise sheet 4

http://staff.aub.edu.lb/~nm116/teaching/2017/math261/index.html

Version: October 2, 2018

Answers are due for Wednesday 10 October, 11AM.

The use of calculators is allowed.

Exercise 4.1: CRT (40 pts)

Find all $x \in \mathbb{Z}$ such that $x \equiv 10 \bmod 100$ and $x \equiv 18 \bmod 31$. Simplify your answer.

Exercise 4.2: Eulers (10 pts)

Compute $\phi(261)$ and $\phi(600)$.

Exercise 4.3: Inverse Euler (50 pts)

The goal of this exercise is to find all integers $n \in \mathbb{N}$ such that $\phi(n) = 4$.

1. (5 pts) Recall the value of $\phi(p^v)$ for $p, v \in \mathbb{N}$ and p prime.
2. (10 pts) Using the previous question, prove that if $p^v \mid n$, then $(p-1)p^{v-1} \mid \phi(n)$.
3. (20 pts) Using the previous question, prove that if $\phi(n) = 4$, then n cannot be divisible by a prime $p \geq 7$. Also prove that $3^2,\ 5^2 \nmid n$.
4. (15 pts) Find all n such that $\phi(n) = 4$.

   Hint: Think in terms of the factorisation of n. You should find that there are four such n — but you are required to prove this as part of this question!

The exercise below is not mandatory. It is not worth any points, and is given here for you to practice. The solutions will be made available with the solutions to the other exercises.

Exercise 4.4: More inverse Eulers (0 pts)

This exercise is difficult, but doable. The questions are independent from each other.

1. Using the fact that $2018 = 2 \times 1009$ and that 1009 is prime, prove that there is no $n \in \mathbb{N}$ such that $\phi(n) = 2018$.

   Hint: Suppose 1009 is a factor of $\phi(n)$. Where can this factor come from?

2. Prove that for all $m \in \mathbb{N}$, there are at most finitely many[1] $n \in \mathbb{N}$ such that $\phi(n) = m$.

   Hint: Try to bound the prime factors of n in terms of m.

3. Prove that $\phi(n)$ is even for all $n \geq 3$.

   Hint: Start with the case when n is a prime power.

[1] This means either finitely many or none.