Math 261 — Exercise sheet 2

http://staff.aub.edu.lb/~nm116/teaching/2017/math261/index.html

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Answers are due for Wednesday 26 September, 11AM.

The use of calculators is allowed.

Exercise 2.1: $ax + by$ (20 pts)

1. (10 pts) When $I, J \subset \mathbb{Z}$ are two subsets of $\mathbb{Z}$, we denote by

   $$I + J = \{i + j \mid i \in I, j \in J\}$$

   the set of integers that can be written as the sum of an element of I and of an element of J.

   Prove that if I and J are ideals of $\mathbb{Z}$, then $I + J$ is also an ideal of $\mathbb{Z}$.

   Hint: $i + j + i' + j' = i + i' + j + j'$.

2. (10 pts) Let now $a, b \in \mathbb{N}$. By the previous question, $a\mathbb{Z} + b\mathbb{Z}$ is an ideal, so it is of the form $c\mathbb{Z}$ for some $c \in \mathbb{N}$. Express c in terms of a and b. What is the name of the theorem that we thus recover?

   Hint: If you are lost, write an English sentence describing the set $a\mathbb{Z} + b\mathbb{Z}$.

Exercise 2.2: Min-max (40 pts)

Let $a, b \in \mathbb{N}$. We may write

$$a = \prod_{i=1}^{r} p_i^{v_i}, \quad b = \prod_{i=1}^{r} p_i^{w_i}$$

with the same (pairwise distinct) primes p_i, by allowing $v_i, w_i \geq 0$.

1. (10 pts) Express $\gcd(a, b)$ and $\operatorname{lcm}(a, b)$ in terms of the p_i, v_i, and w_i.

2. (10 pts) Let v, w be two numbers. Prove carefully that $\min(v, w) + \max(v, w) = v + w$ (including the case $v = w$).

3. (10 pts) Deduce from the previous questions a proof of the formula

   $$\gcd(a, b) \operatorname{lcm}(a, b) = ab.$$

4. (10 pts) Find $\operatorname{lcm}(543, 210)$ (you may use results from last week's exercise sheet).

Exercise 2.3: Divisors (40 pts)

The three questions of this exercise are independent of each other. The last one is difficult.

1. (15 pts) Let $N = 1200$. Find the number of positive divisors of N, the sum of these divisors, and the sum of the squares of these divisors.

2. (20 pts) Find an integer M of the form $3^a 5^b$ such that the sum of the positive divisors of M is 33883.

   Hint: $33883 = 31 \times 1093$, *and both factors are prime.*

3. (5 pts) Find an integer L of the form $2^a 3^b$ such that the **product** of the divisors of L is 12^{15}.

   Hint: What are the divisors of L? Can you arrange them in a 2-dimensional array? Count the number of 2's, and deduce that the 2-adic valuation the product of all these divisors is $(b+1)(1+2+3+\cdots+a)$. What about the 3-adic valuation?

The exercise below has been added for practice. It is not mandatory, and not worth any points. The solution will be made available with the solutions to the other exercises.

Exercise 2.4: $\sqrt{n}$ is either an integer or irrational

Let n be a positive integer which is **not a square**, so that $\sqrt{n}$ is not an integer. The goal of this exercise is to prove that $\sqrt{n}$ is *irrational*, i.e. not of the form $\frac{a}{b}$ where a and b are integers.

1. Prove that there exists at least one prime p such that the p-adic valuation $v_p(n)$ is odd.

2. Suppose on the contrary that $\sqrt{n} = \frac{a}{b}$ with $a, b \in \mathbb{N}$; this may be rewritten as $a^2 = nb^2$. Examine the p-adic valuations of both sides of this equation, and derive a contradiction.