Math 261 — Exercise sheet 7

http://staff.aub.edu.lb/~nm116/teaching/2017/math261/index.html

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Answers are due for Monday 20 November, 11AM.

The use of calculators is allowed.

Exercise 7.1: Reduction (20 pts)

1. (10 pts) Find a reduced quadratic form equivalent to the form $22x^2-16xy+3y^2$.
2. (10 pts) Are the forms $2x^2+xy+3y^2$ and $2x^2-xy+3y^2$ equivalent ?

Solution 7.1:

1. Since $a=22>c=3$, we start by applying the transformation $\left[\begin{smallmatrix}0 & -1\\ 1 & 0\end{smallmatrix}\right]$, which gives us
$$3x^2+16xy+22y^2.$$
Now we have $b=16>a=3$, so we apply the transformation $\left[\begin{smallmatrix}1 & -3\\ 0 & 1\end{smallmatrix}\right]$ (since $\lfloor b/2a \rceil = 3$), and we get
$$3x^2-2xy+y^2.$$
This time $a=3>c=1$, so we apply again the transformation $\left[\begin{smallmatrix}0 & -1\\ 1 & 0\end{smallmatrix}\right]$and we get
$$x^2+2xy+3y^2.$$
Since now $b=2>a=1$, we apply $\left[\begin{smallmatrix}1 & 1\\ 0 & 1\end{smallmatrix}\right]$ (where $1=-\lfloor b/2a \rceil$) and this leads us to
$$x^2+2y^2,$$
which is reduced, so we're done.
2. No, since they are both reduced but distinct.

Exercise 7.2: Class numbers (30 pts)

Compute the class number $h(D)$ for

1. (15 pts) $D=-116$,
2. (15 pts) $D=-47$.

Solution 7.2:

1. If (a, b, c) is reduced of discriminant D, we must have $a \leqslant \sqrt{116/3} < \sqrt{40} < 7$. Also, b must be even.

   We apply the method seen in class:

$$\begin{array}{lll}
a=1: & b=0: & (1,0,29) \\
a=2: & b=0: & \text{✗} \\
 & b=2: & (2,2,15) \\
a=3: & b=0: & \text{✗} \\
 & b=\pm 2: & (3,\pm 2,10) \\
a=4: & b=0: & \text{✗} \\
 & b=\pm 2: & \text{✗} \\
 & b=4: & \text{✗} \\
a=5: & b=0: & \text{✗} \\
 & b=\pm 2 & (5,\pm 2,6) \\
 & b=\pm 4: & \text{✗} \\
a=6: & b=0: & \text{✗} \\
 & b=\pm 2: & \xcancel{(6,\pm 2,5)} \text{ (not reduced)} \\
 & b=\pm 4: & \text{✗} \\
 & b=6: & \text{✗}
\end{array}$$

   We find 6 reduced forms, so $h(-116) = 6$.

2. Same thing, except that this time $a \leqslant \sqrt{47/3} < \sqrt{16} = 4$ and b is odd.

$$\begin{array}{lll}
a=1: & b=1: & (1,1,12) \\
a=2: & b=\pm 1: & (2,\pm 1,6) \\
a=3: & b=\pm 1: & (3,\pm 1,4) \\
 & b=3: & \text{✗}
\end{array}$$

   so $h(-47) = 5$.

Exercise 7.3: Primes of the form... (30 pts)

Let $p \in \mathbb{N}$ be prime.

1. (15 pts) Prove that p is of the form $x^2 + 3y^2$ (with $x, y \in \mathbb{Z}$) if and only if $p = 3$ or $p \equiv 1 \pmod 3$.

2. (15 pts) Prove that p is of the form $x^2 + xy + 3y^2$ (with $x, y \in \mathbb{Z}$) if and only if $p = 11$ or $p \equiv 1, 3, 4, 5$ or $9 \pmod{11}$.

Note: You are ***not*** *allowed to use the theorem giving the list of D such that $h(D) = 1$ in this exercise.*

Solution 7.3:

1. The discriminant of $x^2 + 3y^2$ is -12. We know that if $p \mid 2times - 12$, then p is represented by one of the $h(-12)$ reduced forms of discriminant -12 iff. -12 is a square mod p.

One the one hand, when we look for reduced forms of discriminant forms, we find (as in the previous exercise) that $a \leqslant 2$, b is even, and we get

$$\begin{array}{lll} a=1: & b=0: & (1,0,3) \\ a=2: & b=0: & \text{✗} \\ & b=2: & \cancel{(2,2,2)} \text{ (not primitive)} \end{array}$$

so $h(-12) = 1$ and all the forms of discriminant -12 are equivalent to x^2+3y^2.

On the other hand,

$$\left(\frac{-12}{p}\right) = \left(\frac{-1}{p}\right)\left(\frac{3}{p}\right)\left(\frac{4}{p}\right) = (-1)^{p'}(-1)^{p'}\left(\frac{p}{3}\right)1 = \left(\frac{p}{3}\right)$$

for all odd p, so for $p \nmid 2 \times -12$, -12 is a square mod p iff. p is a square mod 3, iff. $p \equiv 1 \pmod 3$ (since $p \not\equiv 0 \pmod 3$).

The remaining cases are $p = 2$ and 3, but 2 is clearly not of the form $x^2 + 3y^2$ whereas 3 clearly is.

2. The discriminant of $x^2 + xy + 3y^2$ is -11, so if $p \neq 2, 11$ then p is represented by one of the forms of discriminant -11 iff. -11 is a square mod p.

   On the one hand, when we compute $h(-11)$ we find $a < 2$ so $a = 1$ and b odd so $b = 1$, so the only possibility is $c = 4$ and so $x^2 + xy + 3y^2$ is the only reduced form of discriminant -11. Thus all forms of discriminant -11 are equivalent to it.

   On the other hand, we compute that

$$\left(\frac{-11}{p}\right) = \left(\frac{p}{11}\right)$$

   just as in the previous question.

   So when $p \neq 2, 11$, we have that p is of the form $x^2 + xy + 3y^2$ iff. p is a (necessarily nonzero) square mod 11, and by trying all values we see that the nonzero squares mod 11 are $1, 3, 4, 5$ and 9 (note: we know that there are $11' = 5$ of them). Besides, we know that $x^2+xy+3y^2 \geqslant (1-1+3)\min(x^2, y^2)$, so 2 is not represented by $x^2+xy+3y^2$, whereas 11 is represented by $x^2+xy+3y^2$ since $(-1)^2 - 1 \times 2 + 3 \times 2^2 = 11$.

Exercise 7.4: Easy cases of the class number 1 problem (20 pts)

1. (10 pts) Let $n \in \mathbb{N}$ be congruent to 1 or 2 mod 4. Prove that $h(-4n) = 1$ if and only if $n < 3$.

   Hint: Imagine that you apply the method seen in class to compute $h(-4n)$. What happens when $n \geqslant 3$?

2. (10 pts) Let $n \in \mathbb{N}$. Prove that if $h(-4n + 1) = 1$, then $n = 2$ or n is odd.

Solution 7.4:

1. Let us treat the case $n < 3$ first. When we compute $h(-4n)$, we get $a \leqslant \sqrt{4n/3} < 2$, so $a = 1$, and $b = 0$ since b is even, so $c = n$; thus $h(-4n) = 1$ is this case.

   Now if $n \geqslant 3$, then we still have the reduced form $x^2 + ny^2$, but now we can also have $a = 2$, and so $b = 0$ or $b = 2$.

$$\begin{array}{lll} a = 1: & b = 0: & (1, 0, n) \\ a = 2: & b = 0: & (2, 0, n/2) \text{ if } n \text{ even} \\ & b = 2: & (2, 2, \frac{n+1}{2}) \text{ if } n \text{ odd} \\ & \vdots & \end{array}$$

   So if $n \equiv 2 \pmod 4$, then $n/2$ is an odd integer, so the form $(2, 0, n/2)$ is primitive; thus $h(-4n) \geqslant 2$. And if $n \equiv 1 \pmod 4$, then $\frac{n+1}{2}$ is an odd integer, so again the form $(2, 2, \frac{n+1}{2})$ is primitive and $h(-4n) \geqslant 2$.

2. Suppose on the contrary that $n \geqslant 4$ is even. Then for $a = b = 1$ we have the form $(1, 1, n)$, whereas for $a = 2$, $b = 1$ we find the forms $(2, 1, n/2)$, which is primitive and reduced since $2 \leqslant n/2$. So we have $h(-4n + 1) \geqslant 2$.