Math 261 — Exercise sheet 1

http://staff.aub.edu.lb/~nm116/teaching/2017/math261/index.html

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Answers are due for Monday 18 September, 11AM.

The use of calculators is allowed.

Exercise 1.1

Use Euclid's algorithm to prove that 2017 and 261 are coprime, and to find integers u and v such that $2017u + 261v = 1$.

Exercise 1.2

1. Factor 261 into primes. Make sure to prove that you factorization is complete, i.e. that the factors you find are prime.
2. Deduce the number of divisors of 261, and the sum of these divisors.
3. Do the same computations with 6000 instead of 261.

Exercise 1.3

Let a, b and c be integers. Suppose that a and b are coprime, and that a and c are coprime. Prove that a and bc are coprime.

Exercise 1.4: Fermat numbers

Let $n \in \mathbb{N}$, and let $N = 2^n + 1$. Prove that if N is prime, then n must be a power of 2.

Hint: use the identity $x^m + 1 = (x+1)(x^{m-1} - x^{m-2} + \cdots - x + 1)$, *which is valid for all* ***odd*** $m \in \mathbb{N}$.

Remark: The Fermat numbers *are the* $F_n = 2^{2^n} + 1$, $n \in \mathbb{N}$. *They are named after the French mathematician Pierre de Fermat, who noticed that* F_0, F_1, F_2, F_3 *and* F_4 *are all prime, and conjectured in 1650 that* F_n *is prime for all* $n \in \mathbb{N}$. *However, this turned out to be wrong: in 1732, the Swiss mathematician Leonhard Euler proved that* $F_5 = 641 \times 6700417$ *is not prime. To this day, no other prime Fermat number has been found; in fact it is unknown if there is any ! This is because* F_n *grows very quickly with* n, *which makes it very difficult to test whether* F_n *is prime, even with modern computers.*

Exercise 1.5: $\sqrt{n}$ is either an integer or irrational

Let n be a positive integer which is **not a square**, so that $\sqrt{n}$ is not an integer. The goal of this exercise is to prove that $\sqrt{n}$ is *irrational*, i.e. not of the form $\frac{a}{b}$ where a and b are integers.

1. Prove that there exists at least one prime p such that the p-adic valuation $v_p(n)$ is odd.

2. Suppose on the contrary that $\sqrt{n} = \frac{a}{b}$ with $a, b \in \mathbb{N}$; this may be rewritten as $a^2 = nb^2$. Examine the p-adic valuations of both sides of this equation, and derive a contradiction.

The exercise below is not mandatory. It is not worth any points, and it is also more difficult than the previous ones. I highly recommend that you try to solve it for practice. The solution will be made available with the solutions to the other exercises.

Exercise 1.6: Perfect numbers

A positive integer n is said to be *perfect* if it agrees with the sum of all of its divisors other than itself; in other words, if $\sigma_1(n) = 2n$. For instance, 6 is a perfect number, because its divisors other than itself are 1, 2 and 3, and $1 + 2 + 3 = 6$.

1. Let a be a positive integer, and let $n = 2^a(2^{a+1} - 1)$. Prove that if $2^{a+1} - 1$ is prime, then n is perfect.

   We now want to prove that all **even** perfect numbers are of this form.

2. Let n be an even number. Why may we find integers a and b such that $n = 2^a b$ and b is odd ?

3. In this question and in the following ones, we suppose that n is an even perfect number. Prove that $(2^{a+1} - 1) \mid b$.

4. Let thus $c \in \mathbb{N}$ be such that $b = (2^{a+1} - 1)c$. Prove that $\sigma_1(b) = b + c$.

5. Deduce that $c = 1$.

6. Conclude that $2^{a+1} - 1$ is prime.

7. Find two even perfect numbers (apart from 6).

Remarks: Prime numbers of the form $2^a - 1$ *are called* Mersenne primes *after Marin Mersenne (French, 17th century). Not all numbers of the form* $2^a - 1$ *are prime though; in fact, it is not very difficult to show that if* $2^a - 1$*, then* a *is also prime. However this condition is not sufficient, as the counter-example* $2^11 - 1 = 23 \times 89$ *shows. In fact, as of today, only 49 primes* a *such that* $2^a - 1$ *is prime are known. As a result, only 49 Mersenne primes, and so only 49 even perfect numbers, are known. It is conjectured that there exist infinitely many Mersenne primes, and so infinitely many even perfect numbers, but this has never been proved. As for odd perfect numbers, if is unknown if any exist.*