442C Banach algebras 2009–10

1 Introduction to Banach algebras

1.1 Definitions and examples

Let us adopt the convention that all vector spaces and Banach spaces are over the field of complex numbers.

1.1.1 Definition. A *Banach algebra* is a vector space A such that

(i). A is an algebra: it is equipped with an associative product $(a, b) \mapsto ab$ which is linear in each variable,

(ii). A is a Banach space: it has a norm $\|\cdot\|$ with respect to which it is complete, and

(iii). A is a normed algebra: we have $\|ab\| \le \|a\|\,\|b\|$ for $a, b \in A$.

If the product is commutative, so that $ab = ba$ for all $a, b \in A$, then we say that A is an *abelian* Banach algebra.

1.1.2 Remark. The inequality $\|ab\| \le \|a\|\,\|b\|$ ensures that the product is continuous as a map $A \times A \to A$.

1.1.3 Examples. (i). Let X be a topological space. We write $BC(X)$ for the set of bounded continuous functions $X \to \mathbb{C}$. Recall from [FA 1.7.2] that $BC(X)$ is a Banach space under the pointwise vector space operations and the uniform norm, which is given by

$$\|f\| = \sup_{x \in X} |f(x)|, \quad f \in BC(X).$$

The pointwise product

$$(fg)(x) = f(x)g(x), \quad f, g \in BC(X),\ x \in X$$

turns $BC(X)$ into an abelian Banach algebra. Indeed, the product is clearly commutative and associative, it is linear in f and g, and

$$\|fg\| = \sup_{x \in X} |f(x)|\,|g(x)| \le \sup_{x_1 \in X} |f(x_1)| \cdot \sup_{x_2 \in X} |g(x_2)| = \|f\|\,\|g\|.$$

If X is a compact space, then every continuous function $X \to \mathbb{C}$ is bounded. For this reason, we will write $C(X)$ instead of $BC(X)$ if X is compact.

(ii). If A is a Banach algebra, a *subalgebra* of A is a linear subspace $B \subseteq A$ such that $a, b \in B \implies ab \in B$. If B is a closed subalgebra of a Banach algebra A, then B is complete, so it is a Banach algebra (under the same operations and norm as A). We then say that B is a *Banach subalgebra* of A.

(iii). Let $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$ and $\overline{\mathbb{D}} = \{z \in \mathbb{C} : |z| \le 1\}$. The *disc algebra* is the following Banach subalgebra of $C(\overline{\mathbb{D}})$:

$$A(\overline{\mathbb{D}}) = \{f \in C(\overline{\mathbb{D}}) : f \text{ is analytic on } \mathbb{D}\}.$$

(iv). The set $C_0(\mathbb{R})$ of continuous functions $f : \mathbb{R} \to \mathbb{C}$ such that

$$\lim_{x\to\infty} f(x) = \lim_{x\to-\infty} f(x) = 0.$$

This is a Banach subalgebra of $BC(\mathbb{R})$.

(v). Let $\ell^1(\mathbb{Z})$ denote the vector space of complex sequences $(a_n)_{n\in\mathbb{Z}}$ indexed by $\mathbb{Z}$ such that $\|a\| = \sum_{n\in\mathbb{Z}} |a_n| < \infty$. This is a Banach space by [FA 1.7.10]. We define a product $*$ such that if $a = (a_n)_{n\in\mathbb{Z}}$ and $b = (b_n)_{n\in\mathbb{Z}}$ are in $\ell^1(\mathbb{Z})$ then the nth entry of $a * b$ is

$$(a * b)_n = \sum_{m\in\mathbb{Z}} a_m b_{n-m}.$$

This series is absolutely convergent, and $a * b \in \ell^1(\mathbb{Z})$, since

$$\begin{aligned}\|a * b\| &= \sum_n |(a * b)_n| = \sum_n \Big|\sum_m a_m b_{n-m}\Big| \\ &\le \sum_{m,n} |a_m|\,|b_{n-m}| = \sum_m |a_m| \sum_n |b_{m-n}| = \|a\|\,\|b\| < \infty.\end{aligned}$$

It is an exercise to show that $*$ is commutative, associative and linear in each variable, so it turns $\ell^1(\mathbb{Z})$ into an abelian Banach algebra.

(vi). If X is a Banach space, let $\mathcal{B}(X)$ denote the set of all bounded linear operators $T : X \to X$ with the operator norm

$$\|T\| = \sup_{x\in X,\, \|x\|\le 1} \|Tx\|.$$

By [FA 3.3], $\mathcal{B}(X)$ is a Banach space. Define a product on $\mathcal{B}(X)$ by $ST = S \circ T$. This is clearly associative and bilinear, and if $x \in X$ with $\|x\| \le 1$ then

$$\|(ST)x\| = \|S(Tx)\| \le \|S\|\,\|Tx\| \le \|S\|\,\|T\|$$

so $\|ST\| \le \|S\|\,\|T\|$. Hence $\mathcal{B}(X)$ is a Banach algebra. If $\dim X > 1$ then $\mathcal{B}(X)$ is not abelian.

1.2 Invertibility

1.2.1 Definition. A Banach algebra A is *unital* if A contains an identity element of norm 1; that is, an element $1 \in A$ such that $1a = a1 = a$ for all $a \in A$, and $\|1\| = 1$. We call 1 the *unit* of A. We sometimes write $1 = 1_A$ to make it clear that 1 is the unit of A.

If A is a unital Banach algebra and $B \subseteq A$, we say that B is a *unital subalgebra* of A if B is a subalgebra of A which contains the unit of A.

1.2.2 Examples. Of the Banach algebras in Example 1.1.3, only $C_0(\mathbb{R})$ is non-unital. Indeed, it is easy to see that no $f \in C_0(\mathbb{R})$ is an identity element for $C_0(\mathbb{R})$. On the other hand the constant function taking the value 1 is the unit for $BC(X)$, and $A(\overline{\mathbb{D}})$ is a unital subalgebra of $C(\overline{\mathbb{D}})$. Also, the identity operator $I\colon X \to X$, $x \to x$ is the unit for $\mathcal{B}(X)$, and is it not hard to check that the sequence $(\delta_{n,0})_{n\in\mathbb{Z}}$ is the unit for $\ell^1(\mathbb{Z})$.

1.2.3 Remarks. (i). An algebra can have at most one identity element.

(ii). If $(A, \|\cdot\|)$ is a non-zero Banach algebra with an identity element then we can define an norm $|\cdot|$ on A under which it is a unital Banach algebra such that $|\cdot|$ is equivalent to $\|\cdot\|$, meaning that there are constants $m, M \geq 0$ such that

$$m|a| \leq \|a\| \leq M|a| \quad \text{for all } a \in A.$$

For example, we could take $|a| = \|L_a\colon A \to A\|$ where L_a is the linear operator $L_a(b) = ab$ for $a, b \in A$.

1.2.4 Definition. Let A be a Banach algebra with unit 1. An element $a \in A$ is *invertible* if $ab = 1 = ba$ for some $b \in A$. It's easy to see that b is then unique; we call b the *inverse* of a and write $b = a^{-1}$.

We write $\operatorname{Inv} A$ for the set of invertible elements of A.

1.2.5 Remarks. (i). $\operatorname{Inv} A$ forms a group under multiplication.

(ii). If $a \in A$ is left invertible and right invertible so that $ba = 1$ and $ac = 1$ for some $b, c \in A$, then a is invertible.

(iii). If $a = bc = cb$ then a is invertible if and only if b and c are invertible. It follows by induction that if $b_1, \dots, b_n$ are commuting elements of A (meaning that $b_i b_j = b_j b_i$ for $1 \leq i, j \leq n$) then $b_1 b_2 \dots b_n$ is invertible if and only if $b_1, \dots, b_n$ are all invertible.

(iv). The commutativity hypothesis is essential in (iii). For example, if $(e_n)_{n\geq 1}$ is an orthonormal basis of a Hilbert space H and $S \in \mathcal{B}(H)$ is defined by $Se_n = e_{n+1}$, $n \geq 1$ and S^* is the adjoint of S (see [FA 4.18]) then S^*S is invertible although S^* and S are not.

1.2.6 Examples. (i). If X is a compact topological space then

$$\operatorname{Inv} C(X) = \{f \in C(X) \colon f(x) \neq 0 \text{ for all } x \in X\}.$$

Indeed, if $f(x) \neq 0$ for all $x \in X$ then we can define $g \colon X \to \mathbb{C}$, $x \mapsto f(x)^{-1}$. The function g is then continuous [why?] with $fg = 1$. Conversely, if $x \in X$ and $f(x) = 0$ then $fg(x) = f(x)g(x) = 0$ so $fg \neq 1$ for all $g \in C(X)$, so f is not invertible.

(ii). If X is a Banach space then

$$\operatorname{Inv} \mathcal{B}(X) \subseteq \big\{T \in \mathcal{B}(X) \colon \ker T = \{0\}\big\}.$$

Indeed, if $\ker T \neq \{0\}$ then T is not injective, so cannot be invertible.

If X is finite-dimensional and $\ker T = \{0\}$ then by linear algebra, T is surjective, so T is an invertible linear map. Since X is finite-dimensional, the linear map T^{-1} is bounded, so T is invertible in $\mathcal{B}(X)$. Hence

$$\operatorname{Inv} \mathcal{B}(X) = \big\{T \in \mathcal{B}(X) \colon \ker T = \{0\}\big\} \quad \text{if } \dim X < \infty.$$

On the other hand, if X is infinite-dimensional then we generally have $\operatorname{Inv} \mathcal{B}(X) \subsetneq \big\{T \in \mathcal{B}(X) \colon \ker T = \{0\}\big\}$. For example, let $X = H$ be an infinite-dimensional Hilbert space with orthonormal basis $(e_n)_{n \geq 1}$. Consider the operator $T \in \mathcal{B}(H)$ defined by

$$Te_n = \tfrac{1}{n} e_n, \quad n \geq 1.$$

It is easy to see that $\ker T = \{0\}$. However, T is not invertible. Indeed, if $S \in \mathcal{B}(H)$ with $ST = I$ then $Se_n = S(nTe_n) = nSTe_n = ne_n$, so

$$\|Se_n\| = n \to \infty \text{ as } n \to \infty$$

and so S is not bounded, which is a contradiction.

1.2.7 Theorem. *Let A be a Banach algebra with unit 1. If $a \in A$ with $\|a\| < 1$ then $1 - a \in \operatorname{Inv} A$ and*

$$(1-a)^{-1} = \sum_{n=0}^{\infty} a^n.$$

Proof. Since $\|a^n\| \leq \|a\|^n$ and $\|a\| < 1$, the series $\sum_{n=0}^{\infty} a^n$ is absolutely convergent and so convergent by [FA 1.7.8], say to $b \in A$. Let b_n be the nth partial sum of this series and note that

$$b_n(1-a) = (1-a)b_n = (1-a)(1 + a + a^2 + \cdots + a^n) = 1 - a^{n+1} \to 1$$

as $n \to \infty$. So $b(1-a) = (1-a)b = 1$ and so $b = (1-a)^{-1}$. $\square$

1.2.8 Corollary. Inv A *is an open subset of* A.

Proof. Let $a \in \operatorname{Inv} A$ and let $r_a = \|a^{-1}\|^{-1}$. We claim that the open ball $B(a, r_a) = \{b \in A \colon \|a - b\| < r_a\}$ is contained in Inv A; for if $b \in B(a, r_a)$ then $\|a - b\| < r_a$ and

$$b = (a - (a - b))a^{-1}a = (1 - (a - b)a^{-1})a.$$

Since $\|(a - b)a^{-1}\| < r_a\|a^{-1}\| < 1$, the element $1 - (a - b)a^{-1}$ is invertible by Theorem 1.2.7. Hence b is the product of two invertible elements, so is invertible. This shows that every element of Inv A may be surrounded by an open ball which is contained in Inv A, hence Inv A is open. $\square$

1.2.9 Corollary. *The map* $\theta\colon \operatorname{Inv} A \to \operatorname{Inv} A,\ a \mapsto a^{-1}$ *is a homeomorphism.*

Proof. Since $(a^{-1})^{-1} = a$, the map θ is a bijection with $\theta = \theta^{-1}$. So we only need to show that θ is continuous.

If $a \in \operatorname{Inv} A$ and $b \in \operatorname{Inv} A$ with $\|a - b\| < \frac{1}{2}\|a^{-1}\|^{-1}$ then using the triangle inequality and the identity

$$a^{-1} - b^{-1} = a^{-1}(b - a)b^{-1} \tag{$\bigstar$}$$

we have

$$\|b^{-1}\| \le \|a^{-1} - b^{-1}\| + \|a^{-1}\| \le \|a^{-1}\|\,\|a - b\|\,\|b^{-1}\| + \|a^{-1}\| \le \tfrac{1}{2}\|b^{-1}\| + \|a^{-1}\|,$$

so $\|b^{-1}\| \le 2\|a^{-1}\|$. Using ($\bigstar$) again, we have

$$\|\theta(a) - \theta(b)\| = \|a^{-1} - b^{-1}\| \le \|a^{-1}\|\,\|a - b\|\,\|b^{-1}\| < 2\|a^{-1}\|^2\|a - b\|,$$

which shows that θ is continuous at a. $\square$

1.3 The spectrum

1.3.1 Definition. Let A be a unital Banach algebra and let $a \in A$. The *spectrum* of a in A is

$$\sigma(a) = \sigma_A(a) = \{\lambda \in \mathbb{C} \colon \lambda 1 - a \notin \operatorname{Inv} A\}.$$

We will often write λ instead of $\lambda 1$ for $\lambda \in \mathbb{C}$.

1.3.2 Examples. (i). We have $\sigma(\lambda 1) = \{\lambda\}$ for any $\lambda \in \mathbb{C}$.

(ii). Let X be a compact topological space. If $f \in C(X)$ then

$$\sigma(f) = f(X) = \{f(x) \colon x \in X\}.$$

Indeed,

$$\begin{aligned}
\lambda \in \sigma(f) &\iff \lambda 1 - f \notin \operatorname{Inv} C(X) \\
&\iff (\lambda 1 - f)(x) = 0 \text{ for some } x \in X, \text{ by Example 1.2.6(i)} \\
&\iff \lambda = f(x) \text{ for some } x \in X \\
&\iff \lambda \in f(X).
\end{aligned}$$

(iii). If X is a finite-dimensional Banach space and $T \in \mathcal{B}(X)$ then

$$\sigma(T) = \{\lambda \in \mathbb{C} \colon \lambda \text{ is an eigenvalue of } T\}.$$

Indeed,

$$\begin{aligned}
\lambda \in \sigma(T) &\iff T - \lambda \notin \operatorname{Inv} \mathcal{B}(X) \\
&\iff \ker(\lambda I - T) \neq \{0\} \text{ by Example 1.2.6(ii)} \\
&\iff (\lambda I - T)(x) = 0 \text{ for some nonzero } x \in X \\
&\iff Tx = \lambda x \text{ for some nonzero } x \in X \\
&\iff \lambda \text{ is an eigenvalue of } T.
\end{aligned}$$

If X is an infinite-dimensional Banach space, then the same argument shows that $\sigma(T)$ contains the eigenvalues of T, but generally this inclusion is strict.

We will need the following algebraic fact later on.

1.3.3 Proposition. *Let A be a unital Banach algebra and let $a, b \in A$.*

(i). If $1 - ab \in \operatorname{Inv} A$ then $1 - ba \in \operatorname{Inv} A$, and

$$(1 - ba)^{-1} = 1 + b(1 - ab)^{-1}a.$$

(ii). $\sigma(ab) \setminus \{0\} = \sigma(ba) \setminus \{0\}$.

Proof. Exercise. □

To show that the spectrum is always non-empty, we will use a vector-valued version of Liouville's theorem:

1.3.4 Lemma. *Let X be a Banach space and suppose that $f\colon \mathbb{C} \to X$ is an entire function in the sense that $\frac{f(\mu)-f(\lambda)}{\mu-\lambda}$ converges in X as $\mu \to \lambda$, for every $\lambda \in \mathbb{C}$. If f is bounded then f is constant.*

Proof. Given a continuous linear functional $\varphi \in X^*$, let $g = \varphi \circ f\colon \mathbb{C} \to \mathbb{C}$. Since $\frac{g(\mu)-g(\lambda)}{\mu-\lambda} = \varphi(\frac{f(\mu)-f(\lambda)}{\mu-\lambda})$ and $|g(\lambda)| \le \|g\|\,\|f(\lambda)\|$, the function g is entire and bounded. By Liouville's theorem it is constant, so $\varphi(f(\lambda)) = \varphi(f(\mu))$ for all $\varphi \in X^*$ and $\lambda, \mu \in \mathbb{C}$. By the Hahn-Banach theorem (see [FA 3.8]), $f(\lambda) = f(\mu)$ for all $\lambda, \mu \in \mathbb{C}$. So f is constant. $\square$

1.3.5 Theorem. *Let A be a unital Banach algebra. If $a \in A$ then $\sigma(a)$ is a non-empty compact subset of $\mathbb{C}$ with $\sigma(a) \subseteq \{\lambda \in \mathbb{C}\colon |\lambda| \le \|a\|\}$.*

Proof. The map $i\colon \mathbb{C} \to A$, $\lambda \mapsto \lambda - a$ is continuous and

$$\sigma(a) = \{\lambda \in \mathbb{C}\colon i(\lambda) \notin \operatorname{Inv} A\} = \mathbb{C} \setminus i^{-1}(\operatorname{Inv} A).$$

Since $\operatorname{Inv} A$ is open by Corollary 1.2.8 and i is continuous, $i^{-1}(\operatorname{Inv} A)$ is open and so its complement $\sigma(a)$ is closed.

If $|\lambda| > \|a\|$ then $\lambda - a = \lambda(1 - \lambda^{-1}a)$ and $\|\lambda^{-1}a\| = |\lambda|^{-1}\|a\| < 1$ so $\lambda - a$ is invertible by Theorem 1.2.7, so $\lambda \notin \sigma(a)$. Hence $\sigma(a) \subseteq \{\lambda \in \mathbb{C}\colon |\lambda| \le \|a\|\}$. In particular, $\sigma(a)$ is bounded as well as closed, so $\sigma(a)$ is a compact subset of $\mathbb{C}$.

Finally, we must show that $\sigma(a) \ne \emptyset$. If $\sigma(a) = \emptyset$ then the map

$$R\colon \mathbb{C} \to A, \quad \lambda \mapsto (\lambda - a)^{-1}$$

is well-defined. It not hard to show using (★) that

$$\frac{R(\mu) - R(\lambda)}{\mu - \lambda} = -R(\lambda)R(\mu) \quad \text{for } \lambda, \mu \in \mathbb{C} \text{ with } \lambda \ne \mu.$$

Corollary 1.2.9 shows that R is continuous, so we conclude that R is an entire function (with derivative $R'(\lambda) = -R(\lambda)^2$).

Now $\|R(\lambda)\| = \|(\lambda - a)^{-1}\| = |\lambda|^{-1}\|(1 - \lambda^{-1}a)^{-1}\|$ and $1 - \lambda^{-1}a \to 1$ as $|\lambda| \to \infty$ so, by Corollary 1.2.9, $(1 - \lambda^{-1}a)^{-1} \to 1$. Hence $\|R(\lambda)\| \to 0$ as $|\lambda| \to \infty$.

Hence R is a bounded entire function, so it is constant by Lemma 1.3.4; since $R(\lambda) \to 0$ as $|\lambda| \to \infty$ we have $R(\lambda) = 0$ for all $\lambda \in \mathbb{C}$. This is a contradiction since $R(\lambda)$ is invertible for any $\lambda \in \mathbb{C}$. $\square$

The next result says that $\mathbb{C}$ is essentially the only unital Banach algebra which is also a field.

1.3.6 Corollary (The Gelfand-Mazur theorem)**.** *If A is a unital Banach algebra in which every non-zero element is invertible then $A = \mathbb{C}1_A$.*

Proof. Let $a \in A$. Since $\sigma(a) \neq \emptyset$ there is some $\lambda \in \sigma(a)$. Now $\lambda 1 - a \notin \operatorname{Inv} A$, so $\lambda 1 - a = 0$ and $a = \lambda 1 \in \mathbb{C}1$. $\square$

1.3.7 Definition. If a is an element of a unital Banach algebra and $p \in \mathbb{C}[z]$ is a complex polynomial, say $p = \lambda_0 + \lambda_1 z + \cdots + \lambda_n z^n$ where $\lambda_0, \lambda_1, \ldots, \lambda_n$ are complex numbers, then we write

$$p(a) = \lambda_0 1 + \lambda_1 a + \cdots + \lambda_n a^n.$$

1.3.8 Theorem (The spectral mapping theorem for polynomials)**.** *If p is a complex polynomial and a is an element of a unital Banach algebra then*

$$\sigma(p(a)) = p(\sigma(a)) = \{p(\lambda)\colon \lambda \in \sigma(a)\}.$$

Proof. If p is a constant then this immediate since $\sigma(\lambda 1) = \{\lambda\}$. Suppose that $n = \deg p \geq 1$ and let $\mu \in \mathbb{C}$. Since $\mathbb{C}$ is algebraically closed, we can write

$$\mu - p = C(\lambda_1 - z)\ldots(\lambda_n - z)$$

for some $C, \lambda_1, \ldots, \lambda_n \in \mathbb{C}$. Then

$$\mu - p(a) = C(\lambda_1 - a)\ldots(\lambda_n - a)$$

and the factors $\lambda_i - a$ all commute. So

$$\begin{aligned}
\mu \in \sigma(p(a)) &\iff \mu - p(a) \text{ is not invertible}\\
&\iff \text{some } \lambda_i - a \text{ is not invertible (by Remark 1.2.5(iii))}\\
&\iff \text{some } \lambda_i \text{ is in } \sigma(a)\\
&\iff \sigma(a) \text{ contains a root of } \mu - p\\
&\iff \mu = p(\lambda) \text{ for some } \lambda \in \sigma(a).
\end{aligned}$$ $\square$

1.3.9 Definition. Let A be a unital Banach algebra. The *spectral radius* of an element $a \in A$ is

$$r(a) = r_A(x) = \sup_{\lambda \in \sigma_A(a)} |\lambda|.$$

1.3.10 Remark. We have $r(a) \leq \|a\|$ by Theorem 1.3.5.

1.3.11 Examples. (i). If X is a compact topological space and $f \in C(X)$, then

$$r(f) = \sup_{\lambda \in \sigma_A(f)} |\lambda| = \sup_{\lambda \in f(X)} |\lambda| = \|f\|.$$

(ii). To see that strict inequality is possible, take $X = \mathbb{C}^2$ with the usual Hilbert space norm and let $T \in \mathcal{B}(X)$ be the operator with matrix $\left(\begin{smallmatrix} 0 & 1 \\ 0 & 0 \end{smallmatrix}\right)$. Since $\det(T - \lambda I) = \lambda^2$, the only eigenvalue of T is 0 and so $\sigma(T) = \{0\}$ by Example 1.3.2(iii). Hence $r(T) = 0 < 1 = \|T\|$.

1.3.12 Theorem (The spectral radius formula)**.** *The spectral radius of an element of a unital Banach algebra is given by*

$$r(a) = \lim_{n\to\infty} \|a^n\|^{1/n} = \inf_{n\geq 1} \|a^n\|^{1/n}.$$

Proof. If $\lambda \in \sigma(a)$ and $n \geq 1$ then $\lambda^n \in \sigma(a^n)$ by Theorem 1.3.8. So $|\lambda|^n \leq \|a^n\|$ by Theorem 1.3.5, hence $|\lambda| \leq \|a^n\|^{1/n}$ and so $r(a) \leq \inf_{n\geq 1} \|a^n\|^{1/n}$.

Consider the function

$$S\colon \{\lambda \in \mathbb{C}\colon |\lambda| < 1/r(a)\} \to A, \quad \lambda \mapsto (1 - \lambda a)^{-1}.$$

Observe that for $|\lambda| < 1/r(a)$ we have $r(\lambda a) = |\lambda| r(a) < 1$ by Theorem 1.3.8, so $1 - \lambda a$ is invertible and $S(\lambda)$ is well-defined. We can argue as in the proof of Theorem 1.3.5 to see that S is holomorphic. By Theorem 1.2.7 we have $S(\lambda) = \sum_{n=0}^{\infty} \lambda^n a^n$ for $|\lambda| < 1/\|a\|$. If $\varphi \in A^*$ with $\|\varphi\| = 1$ then the complex-valued function $f = \varphi \circ S$ is given by the power series $f(\lambda) = \sum_{n=0}^{\infty} \varphi(a^n)\lambda^n$ for $|\lambda| < 1/\|a\|$. Moreover, f is holomorphic for $|\lambda| < 1/r(a)$, so this power series converges to $f(\lambda)$ for $|\lambda| < 1/r(a)$. Hence for $R > r(a)$ we have

$$\varphi(a^n) = \frac{1}{2\pi i} \int_{|\lambda|=1/R} \frac{f(\lambda)}{\lambda^{n+1}}\, d\lambda$$

and we obtain the estimate

$$|\varphi(a^n)| \leq \frac{1}{2\pi} \cdot \frac{2\pi}{R} \cdot R^{n+1} \cdot \sup_{|\lambda|=1/R} |\varphi(S(\lambda))| \leq R^n M(R)$$

where $M(R) = \sup_{|\lambda|=1/R} \|S(\lambda)\|$, which is finite by the continuity of S on the compact set $\{\lambda \in \mathbb{C}\colon |\lambda| = 1/R\}$. Since $S(\lambda) \neq 0$ for any λ in the domain of S, we have $M(R) > 0$. Hence

$$\limsup_{n\geq 1} \|a^n\|^{1/n} \leq \limsup_{n\geq 1} RM(R)^{1/n} = R$$

whenever $R > r(a)$. We conclude that

$$r(a) \le \inf_{n\ge 1} \|a^n\|^{1/n} \le \liminf_{n\ge 1} \|a^n\|^{1/n} \le \limsup_{n\ge 1} \|a^n\|^{1/n} \le r(a)$$

and the result follows. □

1.3.13 Corollary. *If A is a unital Banach algebra and B is a closed unital subalgebra of A then $r_A(b) = r_B(b)$ for all $b \in B$.*

Proof. The norm of an element of B is the same whether we measure it in B or in A. By the spectral radius formula, $r_A(b) = \lim_{n\ge 1} \|b^n\|^{1/n} = r_B(b)$. □

While the spectral radius of an element of a Banach algebra does not depend if we compute it in a subalgebra, the spectrum itself can change. We explore this in the next few results.

Suppose that A is a unital Banach algebra and B is a unital Banach subalgebra of A. If an element of b is invertible in B, then it is invertible in A; so $\operatorname{Inv} B \subseteq B \cap \operatorname{Inv} A$. However, this inclusion may be strict, as the following example shows.

1.3.14 Example. Recall that $A(\overline{\mathbb{D}})$ is the disc algebra of continuous functions $\overline{\mathbb{D}} \to \mathbb{C}$ which are holomorphic on $\mathbb{D}$. Note that, by the maximum modulus principle, $\sup_{z\in\overline{\mathbb{D}}} |f(z)| = \sup_{\theta\in[0,2\pi)} |f(e^{i\theta})|$. Hence $\|f\| = \|f|_{\mathbb{T}}\|$, and the map $A(\overline{\mathbb{D}}) \to C(\mathbb{T})$, $f \mapsto f|_{\mathbb{T}}$ is a unital isometric isomorphism. So we may identify $A(\overline{\mathbb{D}})$ with $A(\mathbb{T}) = \{f|_{\mathbb{T}} \colon f \in A(\overline{\mathbb{D}})\}$, which is a closed unital subalgebra of $C(\mathbb{T})$.

Consider the function $f(z) = z$ for $z \in \overline{\mathbb{D}}$. This is not invertible in $A(\overline{\mathbb{D}})$ since $f(0) = 0$. Hence $f|_{\mathbb{T}}$ is not invertible in $A(\mathbb{T})$. However, f is invertible in $C(\mathbb{T})$ with inverse $g \colon e^{i\theta} \mapsto e^{-i\theta}$. So $\operatorname{Inv} A(\mathbb{T}) \subsetneq A(\mathbb{T}) \cap \operatorname{Inv} C(\mathbb{T})$.

1.3.15 Definition. If A is a unital Banach algebra then a subalgebra $B \subseteq A$ with $1 \in B$ is said to be *inverse-closed* if $\operatorname{Inv} B = B \cap \operatorname{Inv} A$; that is, if every $b \in B$ which is invertible in A also has $b^{-1} \in B$.

Clearly, if B is an inverse-closed unital subalgebra of A then $\sigma_B(b) = \sigma_A(b)$ for all $b \in B$.

If K is a non-empty compact subset of $\mathbb{C}$ then exactly one of the connected components of $\mathbb{C} \setminus K$ is unbounded. The bounded components of $\mathbb{C} \setminus K$ are called the *holes* of K. If A is a unital Banach algebra and $a \in A$, let us write

$$R_A(a) = \mathbb{C} \setminus \sigma_A(a) = \{\lambda \in \mathbb{C} \colon \lambda - a \in \operatorname{Inv} A\}.$$

This is sometimes called the *resolvent set* of a. Note that the bounded connected components of $R_A(a)$ are precisely the holes of $\sigma_A(a)$.

1.3.16 Theorem. *Let B be a closed subalgebra of a unital Banach algebra A with $1 \in B$. If $b \in B$ then $\sigma_B(b)$ is the union of $\sigma_A(b)$ with zero or more of the holes of $\sigma_A(b)$. In particular, if $\sigma_A(b)$ has no holes then $\sigma_B(b) = \sigma_A(b)$.*

Proof. Since $\operatorname{Inv} B \subseteq \operatorname{Inv} A$ we have $R_B(b) \subseteq R_A(b)$, and so $\sigma_A(b) \subseteq \sigma_B(b)$, for each $b \in B$. We claim that $R_B(b)$ is a relatively clopen subset of $R_A(b)$. Since $\sigma_B(b)$ is closed by Theorem 1.3.5, $R_B(b)$ is open. The map

$$i\colon R_A(b) \to A, \quad \lambda \mapsto (\lambda - b)^{-1}$$

is continuous by Corollary 1.2.9, and

$$R_B(b) = \{\lambda \in R_A(b)\colon i(\lambda) = (\lambda - b)^{-1} \in B\} = i^{-1}(B).$$

Since B is closed, $R_B(b)$ is closed.

If G is a connected component of $R_A(b)$ then $G \cap R_B(b)$ is either $\emptyset$ or G. For otherwise, since $R_B(b)$ is clopen, $G \cap R_B(b)$ and $G \setminus R_B(b)$ would be proper clopen subsets of the connected set G, which is impossible. If G is the unbounded component of $R_A(b)$ then, since $\sigma_B(b)$ is bounded, we must have $G \cap \sigma_B(b) = \emptyset$. The bounded components of $R_A(b)$ are precisely holes of $\sigma_A(b)$. Hence

$$\sigma_B(b) = \sigma_A(b) \cup \bigcup\{G \text{ a hole of } \sigma_A(b)\colon G \cap \sigma_B(b) \neq \emptyset\}.$$

If $\sigma_A(b)$ has no holes then this reduces to $\sigma_B(b) = \sigma_A(b)$. □

1.3.17 Definition. Let A be a Banach algebra. If $S \subseteq A$ then the *commutant* of S in A is

$$S' = \{a \in A\colon ab = ba \text{ for all } b \in S\}.$$

The *bicommutant* of S in A is $S'' = (S')'$.

A set $S \subseteq A$ is *commutative* if $ab = ba$ for all $a, b \in S$. Hence S is commutative if and only if $S \subseteq S'$.

1.3.18 Lemma. *Let A be a Banach algebra. If $T \subseteq S \subseteq A$, then $T' \supseteq S'$. Moreover, $S \subseteq S''$ and $S' = S'''$.*

Proof. Exercise. □

1.3.19 Proposition. *Let A be a unital Banach algebra and let $S \subseteq A$.*

(i). S' is a closed, inverse-closed unital subalgebra of A.

(ii). If S is commutative then so is $B = S''$, and $\sigma_B(b) = \sigma_A(b)$ for all $b \in B$.

Proof. (i) Since multiplication is continuous on A, it is easy to see that the commutant S' is closed. Clearly $1 \in S'$, and using the linearity of multiplication shows that S' is a vector subspace of A, and it is a subalgebra by associativity. If $b \in S' \cap \operatorname{Inv} A$ then $bc = cb$ for all $c \in S$, so $cb^{-1} = b^{-1}c$ for all $c \in S$, so $b^{-1} \in S'$ and S' is inverse-closed.

(ii) We have $S \subseteq S'$, so $S' \supseteq S''$ and $S'' \subseteq S'''$ by Lemma 1.3.18. Hence $B = S''$ is commutative. Moreover, B is an inverse-closed subalgebra of A by (i), so $\sigma_B(b) = \sigma_A(b)$ for all $b \in B$. □

1.3.20 Definition. Let A be a Banach algebra without an identity element. The *unitisation* of A is the Banach algebra $\widetilde{A}$ whose underlying vector space is $A \oplus \mathbb{C}$ with the product $(a, \lambda)(b, \mu) = (ab+\lambda b+\mu a, \lambda\mu)$ and norm $\|(a, \lambda)\| = \|a\| + |\lambda|$ for $a, b \in A$ and $\lambda, \mu \in \mathbb{C}$. Note that $\widetilde{A}$ is then a unital Banach algebra containing A (or, more precisely, $A \times \{0\}$).

1.3.21 Definition. If A has no identity element and $a \in A$, then we define $\sigma_A(a) = \sigma_{\widetilde{A}}(a)$. In this case we have $0 \in \sigma(A)$ for all $a \in A$.

1.3.22 Remark. With this definition, many of the important theorems above apply to non-unital Banach algebras, simply by considering $\widetilde{A}$ instead of A. In particular, it is easy to check that non-unital versions of Theorems 1.3.5, 1.3.8 and 1.3.12 hold.

1.4 Quotients of Banach spaces

Recall that if K is a subspace of a complex vector space X, then the quotient vector space X/K is given by

$$X/K = \{x + K \colon x \in X\}$$
with scalar multiplication $\lambda(x + K) = \lambda x + K, \quad \lambda \in \mathbb{C},\ x \in X$
and vector addition $(x + K) + (y + K) = (x + y) + K, \quad x, y \in X,\ \lambda \in \mathbb{C}.$

The zero vector in X/K is $0 + K = K$.

1.4.1 Definition. If K is a closed subspace of a Banach space X then the *quotient Banach space* X/K is the vector space X/K equipped with the *quotient norm*, defined by

$$\|x + K\| = \inf_{k \in K} \|x + k\|.$$

1.4.2 Proposition. *Let X be a Banach space and let K be a closed vector subspace of X. The quotient norm is a norm on the vector space X/K, with respect to which X/K is complete. Hence the quotient Banach space X/K is a Banach space.*

Proof. To see that the quotient norm is a norm, observe that:

- $\|x + K\| \geq 0$ with equality if and only if $\inf\limits_{k \in K} \|x + k\| = 0$, which is equivalent to x being in the closure of K; since K is closed, this means that $x \in K$ so $x + K = K$, the zero vector of X/K.

- If $\lambda \in \mathbb{C}$ with $\lambda \neq 0$ then

$$\begin{aligned} \|\lambda(x+K)\| &= \inf_{k \in K} \|\lambda x + k\| = |\lambda| \inf_{k \in K} \|x + \lambda^{-1} k\| \\ &= |\lambda| \inf_{k' \in K} \|x + k\| = |\lambda|\, \|x + K\|. \end{aligned}$$

- The triangle inequality holds since $K = \{s + t \colon s, t \in K\}$ and so

$$\begin{aligned} \|(x+K) + (y+K)\| = \|x + y + K\| &= \inf_{k \in K} \|x + y + k\| \\ &= \inf_{s,t \in K} \|x + y + s + t\| \\ &\leq \inf_{s \in K} \|x + s\| + \inf_{t \in K} \|y + t\| \\ &= \|x + K\| + \|y + K\|. \end{aligned}$$

It remains to show that X/K is complete in the quotient norm. For any $x \in X$, it is not hard to see that:

(i) if $\varepsilon > 0$ then there exists $k \in K$ such that $\|x+k\| < \|x+K\| + \varepsilon$; and

(ii) $\|x + K\| \leq \|x\|$ (since $0 \in K$).

Let $x_j \in X$ with $\sum_{j=1}^{\infty} \|x_j + K\| < \infty$. From observation (i), it follows that there exist $k_j \in K$ with $\sum_{j=1}^{\infty} \|x_j + k_j\| < \infty$, so by [FA 1.7.8] the series $\sum_{j=1}^{\infty} x_j + k_j$ converges in X, say to $s \in X$. By observation (ii),

$$\Big\| s + K - \Big(\sum_{j=1}^{n} x_j + K \Big) \Big\| = \Big\| \Big(s - \sum_{j=1}^{n} x_j + k_j \Big) + K \Big\| \leq \Big\| s - \sum_{j=1}^{n} x_j + k_j \Big\| \to 0$$

as $n \to \infty$, so $\sum_{j=1}^{\infty} x_j + k_j$ converges to $s + K$. This shows that every absolutely convergent series in X/K is convergent with respect to the quotient norm, so X/K is complete by [FA 1.7.8]. $\square$

1.5 Ideals, quotients and homomorphisms of Banach algebras

1.5.1 Definition. An *ideal* of a Banach algebra A is a vector subspace I of A such that for all $x \in I$ and $a \in A$ we have $ax \in I$ and $xa \in I$.

1.5.2 Definition. Let I be a closed ideal of a Banach algebra A. The *quotient Banach algebra* A/I is the quotient Banach space A/I equipped with the product $(a+I)(b+I) = ab+I$ for $a, b \in I$.

1.5.3 Theorem. *If I is a closed ideal of a Banach algebra A then A/I is a Banach algebra. If A is abelian then so is A/I. If A is unital then so is A/I, and $1_{A/I} = 1_A + I$.*

Proof. We saw in Proposition 1.4.2 that A/I is a Banach space. Just as for quotient rings, the product is well-defined, since if $a_1 + I = a_2 + I$ and $b_1 + I = b_2 + I$ then $a_1 - a_2 \in I$ and $b_1 - b_2 \in I$, so $a_1(b_1 - b_2) + (a_1 - a_2)b_2 \in I$ and so

$$(a_1b_1 + I) - (a_2b_2 + I) = a_1b_1 - a_2b_2 + I = a_1(b_1 - b_2) + (a_1 - a_2)b_2 + I = I,$$

hence $a_1b_1 + I = a_2b_2 + I$. It is easy to see that this product is linear in each variable.

Let $a, b \in A$. We have

$$\begin{aligned}\|a+I\|\,\|b+I\| &= \inf_{y,z\in I} \|a+y\|\,\|b+z\| \\ &\geq \inf_{y,z\in I} \|(a+y)(b+z)\| \text{ (by 1.1.1(iii) in } A) \\ &= \inf_{y,z\in I} \|ab + (az + yb + yz)\| \\ &\geq \inf_{x\in I} \|ab + x\| \text{ (since } az + by + yz \in I \text{ for all } y, z \in I) \\ &= \|ab + I\| = \|(a+I)(b+I)\|.\end{aligned}$$

Hence the inequality 1.1.1(iii) holds in A/I, and we have shown that A/I is a Banach algebra.

If A is abelian then $(a+I)(b+I) = ab+I = ba+I = (b+I)(a+I)$ for all $a, b \in A$, so A/I is abelian. The proof of the final statement about units is left as an exercise. □

1.5.4 Definition. A *proper ideal* of a Banach algebra A an ideal of A which is not equal to A. A *maximal ideal* of A is a proper ideal such which is not contained in any strictly larger proper ideal of A.

1.5.5 Lemma. *Let A be a unital Banach algebra. If I is an ideal of A, then I is a proper ideal if and only if $I \cap \operatorname{Inv} A = \emptyset$.*

Proof. We have $1 \in \operatorname{Inv} A$, so if $I \cap \operatorname{Inv} A = \emptyset$ then $1 \notin I$, so $I \neq A$ and I is a proper ideal. Conversely, if $I \cap \operatorname{Inv} A \neq \emptyset$, let $b \in I \cap \operatorname{Inv} A$. If $a \in A$ then $a = (ab^{-1})b \in I$, since I is an ideal and $b \in I$. So $I = A$. □

1.5.6 Theorem. *Let A be a unital Banach algebra.*

(i). If I is a proper ideal of A then the closure $\overline{I}$ is also a proper ideal of A.

(ii). Any maximal ideal of A is closed.

Proof. (i) The closure of a vector subspace of A is again a vector subspace. If $a \in A$ and x_n is a sequence in I converging to $x \in \overline{I}$ then $ax_n \to ax$ and $x_n a \to xa$ as $n \to \infty$. Since each ax_n and $x_n a$ is in I, this shows that ax and xa are in $\overline{I}$, which is therefore an ideal of A.

Since I is a proper ideal we have $I \cap \operatorname{Inv} A = \emptyset$ by Lemma 1.5.5. Since $\operatorname{Inv} A$ is open by Corollary 1.2.8, this shows that $\overline{I} \cap \operatorname{Inv} A = \emptyset$ so $\overline{I} \neq A$.

(ii) Let M be a maximal ideal. Since $M \subseteq \overline{M}$ and $\overline{M}$ is a proper ideal by (i), we must have $M = \overline{M}$, so M is closed. □

1.5.7 Remarks. (i). Since we know a few results about ideals of rings, we would like to apply these to ideals of Banach algebras. Any unital Banach algebra A may be viewed as a unital ring R by ignoring the norm and scalar multiplication. However, there is a difference in the definitions: ideals of R are not required to be linear subspaces (since R has no linear structure) whereas ideals of A are. However, the two definitions turn out to be equivalent if A is unital. Indeed, an ideal of the Banach algebra A is clearly an ideal of the ring R. Conversely, if I is an ideal of the ring R then since $\lambda 1 \in R$ for $\lambda \in \mathbb{C}$ we have $\lambda x = \lambda 1 \cdot x \in I$ for all $x \in I$, so I *is* a vector subspace of A with the ideal property. So I is an ideal of A.

(ii). By [FA 2.16], any proper ideal of a unital Banach algebra A is contained in a maximal ideal of A.

1.5.8 Definition. Let A and B be Banach algebras. A *homomorphism* from A to B is a linear map $\theta\colon A \to B$ which is multiplicative in the sense that $\theta(ab) = \theta(a)\theta(b)$ for all $a, b \in A$.

The *kernel* of such a homomorphism θ is the set

$$\ker \theta = \{a \in A\colon \theta(a) = 0\}.$$

If A and B are unital Banach algebras, we say that a homomorphism $\theta\colon A \to B$ is *unital* if $\theta(1_A) = 1_B$.

A bijective homomorphism $\theta\colon A \to B$ is an *isomorphism*. If such an isomorphism exists then the Banach algebras A and B are *isomorphic*. It is easy to see that if θ is an isomorphism then so is θ^{-1}.

1.5.9 Remark. If $\theta\colon A \to B$ is a non-zero homomorphism of Banach algebras then $\ker\theta$ is an ideal of A, which is proper unless $\theta = 0$. If θ is continuous then $\ker\theta$ is a closed ideal of A.

1.5.10 Remark. We usually say that two objects are isomorphic if they have the same structure; that is, if they are the same "up to relabelling". However, if two Banach algebras A and B are isomorphic then this tells us that they have the same structure as algebras, but not necessarily as Banach algebras, since the norms may not be related.

The strongest notion of "the same Banach algebra up to relabelling" is *isometric isomorphism*. Two Banach algebras A and B are isometrically isomorphic if there is an isomorphism $\theta\colon A \to B$ which is also an isometry, meaning that $\|\theta(a)\| = \|a\|$ for all $a \in A$ (compare with [FA 1.3.8]).

1.5.11 Examples. (i). If A is a non-unital Banach algebra then the map $\theta\colon A \to \widetilde{A}$, $a \mapsto (a, 0)$ from Definition 1.3.20 is an isometric homomorphism. Hence $\theta(A)$ is a Banach subalgebra of $\widetilde{A}$ which is isometrically isomorphic to A.

(ii). The map $A(\overline{\mathbb{D}}) \to A(\mathbb{T})$, $f \mapsto f|_{\mathbb{T}}$ from Example 1.3.14 is a unital isometric isomorphism.

1.5.12 Proposition. *Let A and B be unital Banach algebras and let $\theta\colon A \to B$ be a unital homomorphism.*

(i). $\theta(\operatorname{Inv} A) \subseteq \operatorname{Inv} B$, and $\theta(a)^{-1} = \theta(a^{-1})$ for $a \in \operatorname{Inv} A$.

(ii). For all $a \in A$ we have $\sigma_A(a) \supseteq \sigma_B(\theta(a))$.

(iii). If θ is an isomorphism then $\sigma_A(a) = \sigma_B(\theta(a))$ for all $a \in A$.

Proof. (i) If $a \in \operatorname{Inv} A$ then $\theta(a)\theta(a^{-1}) = \theta(aa^{-1}) = \theta(1) = 1$ and $\theta(a^{-1})\theta(a) = \theta(a^{-1}a) = \theta(1) = 1$, so $\theta(a)$ is invertible in B, with inverse $\theta(a^{-1})$.

(ii) If $\lambda \in \sigma_B(\theta(a))$ then $\lambda - \theta(a) = \theta(\lambda - a) \notin \operatorname{Inv} B$ so $\lambda - a \notin \operatorname{Inv} A$ by (i). Hence $\lambda \in \sigma_A(a)$.

(iii) Since θ^{-1} is a homomorphism, by (ii) we have

$$\sigma_A(a) = \sigma_A(\theta^{-1}(\theta(a))) \subseteq \sigma_B(\theta(a)) \subseteq \sigma_A(a),$$

and we have equality. $\square$

2 A topological interlude

2.1 Topological spaces

Recall that a topological space is a set X with a topology: a collection $\mathcal{T}$ of subsets of X, known as *open sets*, such that $\emptyset$ and X are open, and finite intersections and arbitrary unions of open sets are open. We call a set $F \subseteq X$ *closed* if its complement $X \setminus F$ is open.

An open cover C of X is a collection of open subsets of X whose union is X. A finite subcover of C is a finite subcollection whose union still contains X. To say that X is *compact* means that every open cover of X has a finite subcover. Similarly, if $Z \subseteq X$ then an open cover C of Z is a collection of open subsets whose union contains Z. We say that Z is compact if every open cover of Z has a finite subcover. If X is compact, then it is easy to show that any closed subset of X is also compact.

A topological space Y is *Hausdorff* if, for any two distinct points y_1, y_2 in Y, there are disjoint open sets $G_1, G_2 \subseteq Y$ with $y_1 \in G_1$ and $y_2 \in G_2$. It is not hard to show that any compact subset of a Hausdorff space is closed.

If X and Y are topological spaces then a map $\theta\colon X \to Y$ is *continuous* if, for all open sets $G \subseteq Y$, the set $\theta^{-1}(G)$ is open in X. Taking complements, we see that θ is continuous if and only if, for all closed $K \subseteq Y$, the set $\theta^{-1}(K)$ is closed in X. If Z is a compact subset of X and $\theta\colon X \to Y$ is continuous, then $\theta(Z)$ is compact.

A *homeomorphism* from X to Y is a bijection $X \to Y$ which is continuous and has a continuous inverse. If there is a homeomorphism from X to Y, we say that X and Y are *homeomorphic*. If X and Y are homeomorphic topological spaces then all of their topological properties are identical. In particular, X is compact if and only if Y is compact.

Recall that if X is a topological space and $Z \subseteq X$, then the *subspace topology* on Z is defined by declaring the open sets of Z to be the sets $G \cap Z$ for G an open set of X. Then Z is a compact subset of X if and only if Z is a compact topological space (in the subspace topology). Also, it is easy to see that a subspace of a Hausdorff space is Hausdorff.

2.1.1 Lemma. *Let X and Y be topological spaces, and suppose that X is compact and Y is Hausdorff.*

(i). If $\theta\colon X \to Y$ is a continuous bijection, then θ is a homeomorphism onto Y. In particular, Y is compact.

(ii). If $\theta\colon X \to Y$ is a continuous injection, then $\theta(X)$ (with the subspace topology from Y) is homeomorphic to X. In particular, $\theta(X)$ is a compact subset of Y.

Proof. (i) Let K be a closed subset of X. Since X is compact, K is compact. Since θ is continuous, $\theta(K)$ is compact. A compact subset of a Hausdorff space is closed, so $\theta(K)$ is closed. Hence θ^{-1} is continuous, which shows that θ is a homeomorphism. So Y is homeomorphic to the compact space X; so Y is compact.

(ii) The subspace $\theta(X)$ of the Hausdorff space Y is Hausdorff. Let $\widetilde{\theta}\colon X \to \theta(X)$, $x \mapsto \theta(x)$, which is a continuous bijection. By (i), $\widetilde{\theta}$ is a homeomorphism and $\theta(X)$ is compact. □

2.2 Subbases and weak topologies

2.2.1 Definition. If $\mathcal{T}_1$ and $\mathcal{T}_2$ are two topologies on a set X then we say that $\mathcal{T}_1$ is *weaker* than $\mathcal{T}_2$ if $\mathcal{T}_1 \subseteq \mathcal{T}_2$.
[We might also say that $\mathcal{T}_1$ is *smaller*, or *coarser* than $\mathcal{T}_2$].

2.2.2 Definition. Let $(X, \mathcal{T})$ be a topological space. We say that a collection of open sets $\mathcal{S} \subseteq \mathcal{T}$ is a *subbase* for $\mathcal{T}$ if $\mathcal{T}$ is the weakest topology containing $\mathcal{S}$. If the topology $\mathcal{T}$ is understood, we will also say that $\mathcal{S}$ is a subbase for the topological space X.

2.2.3 Remark. Suppose that $\mathcal{T}$ is a topology on X and $\mathcal{S} \subseteq \mathcal{T}$. It is not hard to see that the collection of unions of finite intersections of sets in $\mathcal{S}$ forms a topology on X which is no larger than $\mathcal{T}$. [The empty set is the union of zero sets, and X is the intersection of zero sets, so $\emptyset$ and X are in this collection.] From this, it follows that that following conditions are equivalent:

(i). $\mathcal{S}$ is a subbase for $\mathcal{T}$;

(ii). every set in $\mathcal{T}$ is a union of finite intersections of sets in $\mathcal{S}$;

(iii). a set $G \subseteq X$ is in $\mathcal{T}$ if and only if for every $x \in G$, there exist finitely many sets $S_1, S_2, \ldots, S_n \in \mathcal{S}$ such that

$$x \in S_1 \cap S_2 \cap \cdots \cap S_n \subseteq G.$$

By the equivalence of (i) and (ii), if X, Y are topological spaces and $\mathcal{S}$ is a subbase for X, then a map $f\colon Y \to X$ is continuous if and only if $f^{-1}(S)$ is open for all $S \in \mathcal{S}$.

2.2.4 Proposition. *Let X be a set, let I be an index set and suppose that X_i is a topological space and $f_i\colon X \to X_i$ for each $i \in I$. The collection*

$$\mathcal{S} = \{f_i^{-1}(G)\colon i \in I,\ G \text{ is an open subset of } X_i\}$$

is a subbase for a topology on X, and this is the weakest topology such that f_i is continuous for all $i \in I$.

Proof. Let $\mathcal{T}$ be the collection of all unions of finite intersections of sets from $\mathcal{S}$. Then $\mathcal{T}$ is a topology on X and $\mathcal{S}$ is a subbase for $\mathcal{T}$ by the previous remark. If $\mathcal{T}'$ is any topology on X such that $f_i\colon X \to X_i$ is continuous for all $i \in I$ then by the definition of continuity, $f_i^{-1}(G) \in \mathcal{T}'$ for all open subsets $G \subseteq X_i$, so $\mathcal{S} \subseteq \mathcal{T}'$. Since $\mathcal{T}$ is the weakest topology containing $\mathcal{S}$ this shows that $\mathcal{T} \subseteq \mathcal{T}'$, so $\mathcal{T}$ is the weakest topology such that each f_i is continuous. □

This allows us to introduce the following terminology.

2.2.5 Definition. If X is a set, X_i is a topological space and $f_i\colon X \to X_i$ for $i \in I$ then the weakest topology on X such that f_i is continuous for each $i \in I$ is called the *weak topology* induced by the family $\{f_i\colon i \in I\}$.

2.2.6 Proposition. *Suppose that X is a topological space with the weak topology induced by a family of maps $\{f_i\colon i \in I\}$ where $f_i\colon X \to X_i$ and X_i is a topological space for each $i \in I$.*

If Y is a topological space then a map $g\colon Y \to X$ is continuous if and only if $f_i \circ g\colon Y \to X_i$ is continuous for all $i \in I$.

Proof. The sets $f_i^{-1}(G)$ for $i \in I$ and G an open subset of X_i form a subbase for X, by Proposition 2.2.4. Hence, by Remark 2.2.3,

$$\begin{aligned} g \text{ is continuous} &\iff g^{-1}(f_i^{-1}(G)) \text{ is open for all } i \in I \text{ and open } G \subseteq X_i \\ &\iff (f_i \circ g)^{-1}(G) \text{ is open for all } i \in I \text{ and open } G \subseteq X_i \\ &\iff f_i \circ g \text{ is continuous for all } i \in I. \end{aligned}$$

□

2.2.7 Lemma. *Suppose that X is a topological space with the weak topology induced by a family of mappings $\{f_i\colon X \to X_i\}_{i\in I}$. If $Y \subseteq X$ then the weak topology induced by the family $\{f_i|_Y\colon Y \to X_i\}_{i\in I}$ is the subspace topology on Y.*

Proof. Let $g_i = f_i|_Y$ for $i \in I$. Observe that $g_i^{-1}(G) = f_i^{-1}(G) \cap Y$ for G an open subset of X_i. It is not hard to check that the collection of all sets of this form is a subbase for both the weak topology induced by $\{g_i\}_{i\in I}$ and for the subspace topology on Y. Hence these topologies are equal. □

2.3 The product topology and Tychonoff's theorem

If $\mathcal{S}$ is a collection of open subsets of X, let us say that an open cover is an $\mathcal{S}$-*cover* if every set in the cover is in $\mathcal{S}$.

2.3.1 Theorem (Alexander's subbase lemma)**.** *Let X be a topological space with a subbase $\mathcal{S}$. If every $\mathcal{S}$-cover of X has a finite subcover, then X is compact.*

Proof. Suppose that the hypothesis holds but that X is not compact. Then there is an open cover with no finite subcover; ordering such covers by inclusion we can apply Zorn's lemma [FA 2.15] to find an open cover C of X without a finite subcover that is maximal among such covers.

Note that $C \cap \mathcal{S}$ cannot cover X by hypothesis, so there is some $x \in X$ which does not lie in any set in the collection $C \cap \mathcal{S}$. On the other hand, C does cover X so there is some $G \in C \setminus \mathcal{S}$ with $x \in G$. Since G is open and $\mathcal{S}$ is a subbase, by Remark 2.2.3 we have

$$x \in S_1 \cap \cdots \cap S_n \subseteq G$$

for some $S_1, \ldots, S_n \in \mathcal{S}$. For $i = 1, \ldots, n$ we have $x \in S_i$ so $S_i \notin C$. By the maximality of C, there is a finite subcover C_i of $C \cup \{S_i\}$. Let $D_i = C_i \setminus \{S_i\}$. Then $D = D_1 \cup \cdots \cup D_n$ covers $X \setminus (S_1 \cap \cdots \cap S_n)$. So $D \cup \{G\}$ is a finite subcover of C, which is a contradiction. So X must be compact. $\square$

2.3.2 Definition. Let $\{X_i \colon i \in I\}$ be an indexed collection of sets. Just as in [FA 2.4], we define the *(Cartesian) product* of this collection to be the set

$$\prod_{i \in I} X_i = \{f \colon I \to \bigcup_{i \in I} X_i \colon f(i) \in X_i \text{ for all } i \in I\}.$$

If each X_i is a topological space then the *product topology* on $X = \prod_{i \in I} X_i$ is the weak topology induced by the family $\{\pi_i \colon i \in I\}$ where the map π_i is the "evaluation at i" map

$$\pi_i \colon X \to X_i, \quad f \mapsto f(i).$$

2.3.3 Remark. If $f \in \prod_{i \in I} X_i$ then it is often useful to think of f as the "I-tuple" $(f(i))_{i \in I}$. In this notation, we have

$$\prod_{i \in I} X_i = \{(x_i)_{i \in I} \colon x_i \in X_i \text{ for all } i \in I\}$$

and $\pi_i \colon (x_i)_{i \in I} \mapsto x_i$ is the projection onto the ith coordinate.

2.3.4 Theorem (Tychonoff's theorem)**.**
The product of a collection of compact topological spaces is compact.

Proof. Let X_i be a compact topological space for $i \in I$ and let $X = \prod_{i\in I} X_i$ with the product topology. Consider the collection

$$\mathcal{S} = \{\pi_i^{-1}(G) \colon i \in I,\ G \text{ is an open subset of } X_i\}.$$

By Proposition 2.2.4, $\mathcal{S}$ is a subbase for the topology on X.

Let C be an $\mathcal{S}$-cover of X. For $i \in I$, let $C_i = \{G \subseteq X_i \colon \pi_i^{-1}(G) \in C\}$, which is a collection of open subsets of X_i.

We claim that there is $i \in I$ such that C_i is a cover of X_i. Otherwise, for every $i \in I$ there is some $x_i \in X_i$ not covered by C_i. Consider the map $f \in X$ defined by $f(i) = x_i$. By construction, f does not lie in $\pi_i^{-1}(G)$ for any $i \in I$ and $G \in C_i$. However, since $C \subseteq \mathcal{S}$, every set in C is of this form, so C cannot cover f. This contradiction establishes the claim.

So we can choose $i \in I$ so that C_i covers X_i. Since X_i is compact, there is finite subcover D_i of C_i. But then $\{\pi_i^{-1}(G) \colon G \in D_i\}$ covers X, and this is a finite subcover of C.

This shows that every $\mathcal{S}$-cover of X has a finite subcover. By Theorem 2.3.1, X is compact. □

2.4 The weak* topology

Let X be a Banach space. Recall from [FA 3.2] that the dual space X^* of X is the Banach space of continuous linear functionals $\varphi : X \to \mathbb{C}$, with the norm $\|\varphi\| = \sup\limits_{\|x\|\le 1} |\varphi(x)|$.

2.4.1 Definition. For $x \in X$, let $J_x \colon X^* \to \mathbb{C}$, $\varphi \mapsto \varphi(x)$. The *weak* topology* on X^* is the weak topology induced by the family $\{J_x \colon x \in X\}$.

2.4.2 Remarks. (i). For $x \in X$, the map J_x is simply the canonical image of x in X^{**}. In particular, each J_x is continuous when X^* is equipped with the usual topology from its norm, so the weak* topology is weaker (that is, no stronger) than the norm topology on X^*. In fact, the weak* topology is generally strictly weaker than the norm topology.

(ii). The sets $\{\psi \in X^* \colon |\psi(x) - \varphi(x)| < \varepsilon\}$ for $\varphi \in X^*$, $\varepsilon > 0$ and $x \in X$ form a subbase for the weak* topology.

(iii). By (ii), it is easy to see that X^* with the weak* topology is a Hausdorff topological space.

2.4.3 Theorem (The Banach-Alaoglu theorem)**.** *Let X be a Banach space. The closed unit ball of X^* is compact in the weak* topology.*

Proof. For $x \in X$ let $D_x = \{\lambda \in \mathbb{C} \colon |\lambda| \le \|x\|\}$. Since D_x is closed and bounded, it is a compact topological subspace of $\mathbb{C}$. Let $D = \prod_{x \in X} D_x$ with the product topology. By Tychonoff's theorem 2.3.4, D is a compact topological space.

Let $X_1^* = \{\varphi \in X^* \colon \|\varphi\| \le 1\}$ denote the closed unit ball of X^*, with the subspace topology that it inherits from the weak* topology on X^*. We must show that X_1^* is compact.

If φ is any linear map $X \to \mathbb{C}$, then $\varphi \in X_1^*$ if and only if $|\varphi(x)| \le \|x\|$ for all $x \in X$, i.e. $\varphi(x) \in D_x$ for all $x \in X$. Thus $X_1^* \subseteq D$. Moreover, if $x \in X$ and $\varphi \in X_1^*$ then

$$J_x(\varphi) = \varphi(x) = \pi_x(\varphi),$$

so $J_x|_{X_1^*} = \pi_x|_{X_1^*}$. By Lemma 2.2.7, the topology on X_1^* is equal to the subspace topology when we view it as a subspace of D.

Since D is compact, it suffices to show that X_1^* is a closed subset of D. Now

$$X_1^* = \{\varphi \in D \colon \varphi \text{ is linear}\} = \bigcap_{\substack{\alpha,\beta \in \mathbb{C}, \\ x,y \in X}} \{\varphi \in D \colon \pi_{\alpha x + \beta y}(\varphi) = \alpha \pi_x(\varphi) + \beta \pi_y(\varphi)\}.$$

By the definition of the product topology on D, the maps $\pi_x \colon D \to D_x$ are continuous for each $x \in X$. Linear combinations of continuous functions are continuous, so for $x, y, z \in X$ and $\alpha, \beta \in \mathbb{C}$, the function $\rho = \pi_z - \alpha\pi_x - \beta\pi_y$ is continuous $D \to \mathbb{C}$. Hence $\rho^{-1}(0)$ is closed. Each set in the above intersection is of this form, so X_1^* is closed. $\square$

3 Unital abelian Banach algebras

3.1 Characters and maximal ideals

Let A be a unital abelian Banach algebra.

3.1.1 Definition. A *character* on A is a non-zero homomorphism $A \to \mathbb{C}$; that is, a non-zero linear map $\tau\colon A \to \mathbb{C}$ which satisfies $\tau(ab) = \tau(a)\tau(b)$ for $a, b \in A$. We write $\Omega(A)$ for the set of characters on A.

3.1.2 Example. Let $A = C(X)$ where X is a compact topological space. For each $x \in X$, the map $\varepsilon_x\colon A \to \mathbb{C}$, $f \mapsto f(x)$ is a character on A.

3.1.3 Remark. This definition makes sense even if A is not abelian. However, $\Omega(A)$ is often not very interesting in that case.

For example, if $A = M_n(\mathbb{C})$ and $n > 1$ then $\Omega(A) = \emptyset$. Indeed, it is not hard to show that A is spanned by $\{ab - ba\colon a, b \in A\}$. If $\varphi\colon A \to \mathbb{C}$ is a homomorphism, then $\varphi(ab - ba) = \varphi(a)\varphi(b) - \varphi(b)\varphi(a) = 0$, so $\varphi = 0$ by linearity; hence $\Omega(A) = \emptyset$.

3.1.4 Lemma. *If $\tau \in \Omega(A)$ then τ is continuous. More precisely,*

$$\|\tau\| = \tau(1) = 1.$$

In particular, $\Omega(A)$ is a subset of the closed unit ball of A^*.

Proof. Observe that $\tau(1) = \tau(1^2) = \tau(1)^2$, so $\tau(1) \in \{0, 1\}$. If $\tau(1) = 0$ then $\tau(a) = \tau(a1) = \tau(a)\tau(1) = 0$ for any $a \in A$, so $\tau = 0$. But $\tau \in \Omega(A)$ so $\tau \neq 0$, which is a contradiction. So $\tau(1) = 1$.

We have $\tau(\operatorname{Inv} A) \subseteq \operatorname{Inv} \mathbb{C} = \mathbb{C} \setminus \{0\}$ by 1.5.12(i). For any $a \in A$ we have $\tau(\tau(a)1 - a) = \tau(a)\tau(1) - \tau(a) = 0$, so $\tau(a)1 - a \notin \operatorname{Inv} A$. Hence $\tau(a) \in \sigma(a)$, so $|\tau(a)| \leq \|a\|$ by Theorem 1.3.5 and so $\|\tau\| \leq 1$. Since $|\tau(1)| = 1$ we conclude that $\|\tau\| = 1$.

Any $\tau \in \Omega(A)$ is a linear map $A \to \mathbb{C}$ by definition, and we have shown that it continuous with norm 1. Hence $\Omega(A)$ is contained in the closed unit ball of A^*. $\square$

3.1.5 Definition. The *Gelfand topology* on $\Omega(A)$ is the subspace topology obtained from the weak* topology on A^*.

We will always equip $\Omega(A)$ with this topology.

3.1.6 Theorem. *$\Omega(A)$ is a compact Hausdorff space.*

Proof. We observed in Remark 2.4.2(iii) that the weak* topology is Hausdorff, so $\Omega(A)$ is a Hausdorff space. By Lemma 3.1.4, $\Omega(A)$ is contained in the unit ball of A^*, which is compact in the weak* topology by Theorem 2.4.3. A closed subset of a compact set is compact, so it suffices to show that $\Omega(A)$ is weak* closed in A^*. But

$$\begin{aligned}\Omega(A) &= \{\tau \in A^* \colon \tau(1) = 1,\ \tau(ab) = \tau(a)\tau(b) \text{ for } a, b \in A\} \\ &= \{\tau \in A^* \colon \tau(1) = 1\} \cap \bigcap_{a,b \in A} \{\tau \in A^* \colon \tau(ab) - \tau(a)\tau(b) = 0\} \\ &= J_1^{-1}(1) \cap \bigcap_{a,b\in A} (J_{ab} - J_a \cdot J_b)^{-1}(0).\end{aligned}$$

Each evaluation functional $J_a \colon A^* \to \mathbb{C}$, $\tau \mapsto \tau(a)$ is weak* continuous, so the maps $J_{ab} - J_a \cdot J_b$ are also weak* continuous. Hence the sets in this intersection are all weak* closed, so $\Omega(A)$ is weak* closed. $\square$

3.1.7 Lemma. *Let A be a unital abelian Banach algebra.*

(i). If $\tau \in \Omega(A)$ then $\ker \tau$ is a maximal ideal of A.

(ii). If M is a maximal ideal of A, then the map $\mathbb{C} \to A/M$, $\lambda \mapsto \lambda 1 + M$ is an isometric isomorphism.

Proof. (i) Let $\tau \in \Omega(A)$. Since τ is a non-zero homomorphism, its kernel $I = \ker \tau$ is a proper ideal of A. Suppose that J is an ideal of A with $I \subsetneq J$ and let $a \in J \setminus I$. Then $\tau(a) \neq 0$, so $b = \tau(a)^{-1}a \in J$ and $\tau(b) = 1$. Since $\tau(1) = 1$ by Lemma 3.1.4, we have $1 - b \in I$, so $1 = b + 1 - b \in J$. By Lemma 1.5.5, $J = A$. This shows that I is not contained in any strictly larger proper ideal of A, so I is a maximal ideal of A.

(ii) Let M be a maximal ideal of A. By Theorems 1.5.6(ii) and 1.5.3, A/M is unital Banach algebra with unit $1 + M$. If $a + M$ is a non-zero element of A/M then $a \in A \setminus M$. Let $I = \{ab + m \colon m \in M,\ b \in A\}$. Since A is abelian and M is an ideal, it is easy to see that I is an ideal of A, and $M \subsetneq I$. Since M is a maximal ideal, $I = A$. So $1 \in I$, and $ab + m = 1$ for some $b \in A$ and $m \in M$. Now

$$(a + M)(b + M) = ab + M = ab + m + M = 1 + M,$$

which is the unit of A/M. Hence $b + M = (a + M)^{-1}$ and $a + M$ is invertible in A/M. By the Gelfand-Mazur theorem 1.3.6, $A/M = \mathbb{C}1_{A/M} = \mathbb{C}(1 + M)$.

It is very easy to check that the map $\mathbb{C} \to A/M$, $\lambda \mapsto \lambda 1 + M$ is an isometric homomorphism, and we have just shown that it is surjective. Hence it is an isometric isomorphism. $\square$

3.1.8 Theorem. *Let A be a unital abelian Banach algebra. The mapping*

$$\tau \mapsto \ker \tau$$

is a bijection from $\Omega(A)$ onto the set of maximal ideals of A.

Proof. If $\tau \in \Omega(A)$ then $\ker \tau$ is a maximal ideal of A, by Lemma 3.1.7(i). Hence the mapping is well-defined.

The mapping $\tau \mapsto \ker \tau$ is injective, since if τ_1 and τ_2 are in $\Omega(A)$ with $\ker \tau_1 = \ker \tau_2$, then for any $a \in A$ we have $a - \tau_2(a)1 \in \ker \tau_2 = \ker \tau_1$ so $\tau_1(a - \tau_2(a)1) = 0$, hence $\tau_1(a) = \tau_2(a)$; so $\tau_1 = \tau_2$.

We now show that the mapping is surjective. Let M be a maximal ideal of A, and let $q\colon A \to A/M$, $a \mapsto a + M$ be the corresponding quotient map. Observe that q is a homomorphism and $\ker q = M$. By Lemma 3.1.7(ii), the map $\theta\colon \mathbb{C} \to A/M$, $\lambda \mapsto \lambda 1 + M$ is an isomorphism. Let $\tau = \theta^{-1} \circ q\colon A \to \mathbb{C}$. Since τ is the composition of two homomorphisms, it is a homomorphism, and $\tau(1) = \theta^{-1}(q(1)) = \theta^{-1}(1 + M) = 1$, so $\tau \neq 0$. Hence $\tau \in \Omega(A)$. Since θ is an isomorphism, we have $\ker \tau = \ker q = M$.

This shows that $\tau \mapsto \ker \tau$ is a bijection from $\Omega(A)$ onto the set of all maximal ideals of A. $\square$

3.1.9 Examples. (i). Let X be a compact Hausdorff space. For $x \in X$, the map $\varepsilon_x\colon C(X) \to \mathbb{C}$, $f \mapsto f(x)$ is a nonzero homomorphism, so $\{\varepsilon_x\colon x \in X\} \subseteq \Omega(C(X))$. We claim that we have equality.

For $x \in X$, let $M_x = \ker \varepsilon_x = \{f \in C(X)\colon f(x) = 0\}$, which is a maximal ideal of $C(X)$ by Theorem 3.1.8. Let I be an ideal of $C(X)$. If $I \not\subseteq M_x$ for every $x \in X$, then for each $x \in X$, there is $f_x \in I$ with $f_x \neq 0$. Since I is an ideal, $g_x = |f_x|^2 = \overline{f_x} f_x \in I$, and since g_x is continuous and non-negative with $g_x(x) > 0$, there is an open set U_x with $x \in U_x$ and $g_x(y) > 0$ for all $y \in U_x$. As x varies over X, the open sets U_x cover X. Since X is compact, there is $n \geq 1$ and $x_1, \dots, x_n \in X$ such that $U_{x_1}, \dots, U_{x_n}$ cover X. Let $g = g_{x_1} + \dots + g_{x_n}$. Then $g \in I$ and $g(x) > 0$ for all $x \in X$, so g is invertible in $C(X)$. Hence $I = C(X)$.

This shows that every proper ideal I of $C(X)$ is contained in M_x for some $x \in X$. Let $\tau \in \Omega(C(X))$. Since $\ker \tau$ is a maximal (proper) ideal, we must have $\ker \tau = M_x$ for some $x \in X$, so $\tau = \varepsilon_x$ by Theorem 3.1.8.

Consider the map $\theta\colon X \to \Omega(C(X))$, $x \mapsto \varepsilon_x$. We have just shown that this is surjective. Since X is compact and Hausdorff, $C(X)$ separates the points of X by Urysohn's lemma. Hence if $\varepsilon_x = \varepsilon_y$ then $f(x) = f(y)$ for all $f \in C(X)$, so $x = y$. Hence θ is a bijection.

We claim that θ is a homeomorphism. Indeed, θ is continuous since for $f \in C(X)$ and $x \in X$ we have

$$J_f(\theta(x)) = J_f(\varepsilon_x) = \varepsilon_x(f) = f(x),$$

so $J_f \circ \theta$ (or, more precisely, $J_f|_{\Omega(C(X))} \circ \theta$) is continuous $X \to \mathbb{C}$ for every $f \in C(X)$. By Proposition 2.2.6, θ is continuous. Since X is compact and $\Omega(C(X))$ is Hausdorff, Lemma 2.1.1 shows that θ is a homeomorphism.

(ii). Recall that $A(\overline{\mathbb{D}})$ denotes the disc algebra. If $w \in \mathbb{D}$ then $\varepsilon_w : A(\overline{\mathbb{D}}) \to \mathbb{C}$, $f \mapsto f(w)$ is a character on $A(\overline{\mathbb{D}})$.

Again, we claim that every character arises in this way. To see this, consider the function $z \in A(\overline{\mathbb{D}})$ defined by $z(w) = w$, $w \in \overline{\mathbb{D}}$. If $\tau \in \Omega(A(\overline{\mathbb{D}}))$ then $\tau(1) = 1$ and $|\tau(z)| \leq \|z\| = 1$, so $\tau(z) \in \overline{\mathbb{D}}$. It is not hard to show that the polynomials $\mathbb{D} \to \mathbb{C}$ form a dense unital subalgebra of $A(\overline{\mathbb{D}})$. If $p\colon \mathbb{D} \to \mathbb{C}$ is a polynomial then $p = \lambda_0 1 + \lambda_1 z + \cdots + \lambda_n z^n$ for constants λ_i, so $\tau(p) = \lambda_0 + \lambda_1 \tau(z) + \cdots + \lambda_n \tau(z)^n = p(\tau(z))$. Since the polynomials are dense in $A(\overline{\mathbb{D}})$, this shows that $\tau(f) = f(\tau(z))$ for all $f \in A(\overline{\mathbb{D}})$, so $\tau = \varepsilon_{\tau(z)}$. Hence $\Omega(A(\overline{\mathbb{D}})) = \{\varepsilon_w\colon w \in \overline{\mathbb{D}}\}$. Just as in (i), it is easy to see that the map $w \mapsto \varepsilon_w$ is a homeomorphism $\overline{\mathbb{D}} \to \Omega(A(\overline{\mathbb{D}}))$.

(iii). Recall the abelian Banach algebra $\ell^1(\mathbb{Z})$ with product $*$ from Example 1.1.3(v). For $n \in \mathbb{Z}$, let $e_n = (\delta_{mn})_{m\in\mathbb{Z}}$. Then $e_n \in \ell^1(\mathbb{Z})$, and the linear span of $\{e_n\colon n \in \mathbb{Z}\}$ is dense in $\ell^1(\mathbb{Z})$. Moreover, it is easy to check that $e_n * e_m = e_{n+m}$ for $n, m \in \mathbb{Z}$. In particular, $e_0 * e_m = e_m$, hence e_0 is the unit for $\ell^1(\mathbb{Z})$.

If $\tau \in \Omega(\ell^1(\mathbb{Z}))$ then $\tau(e_0) = 1$. Moreover, $e_n * e_{-n} = e_0$ so $e_n = (e_{-n})^{-1}$ and $|\tau(e_n)| \leq \|e_n\| = 1$ for each $n \in \mathbb{Z}$, so

$$1 \leq |\tau(e_{-n})|^{-1} = |\tau((e_{-n})^{-1})| = |\tau(e_n)| \leq 1.$$

So we have equality. In particular, $|\tau(e_1)| = 1$, so $\tau(e_1) \in \mathbb{T}$. Since $e_n = e_1^n$, we have $\tau(e_n) = \tau(e_1)^n$. Hence if $x \in \ell^1(\mathbb{Z})$ then

$$\tau(x) = \tau\left(\sum_{n\in\mathbb{Z}} x_n e_n\right) = \sum_{n\in\mathbb{Z}} x_n \tau(e_1)^n.$$

So τ is determined by the complex number $\tau(e_1) \in \mathbb{T}$. Conversely, given any $z \in \mathbb{T}$, there is a character $\tau_z \in \Omega(\ell^1(\mathbb{Z}))$ with $\tau_z(e_1) = z$. Indeed, let $A_0 = \operatorname{span}\{e_n\colon n \in \mathbb{Z}\}$, which is a dense subalgebra of $\ell^1(\mathbb{Z})$. Let

$\tau_0 \colon A_0 \to \mathbb{C}$ be the unique linear map such that $\tau_0(e_n) = z^n$ for all $n \in \mathbb{Z}$. If $x, y \in A_0$, then

$$\tau_0(x * y) = \tau_0\Big(\sum_{m,n\in\mathbb{Z}} x_m y_{n-m} e_n\Big) = \sum_{m,n\in\mathbb{Z}} x_m y_{n-m} z^n$$
$$= \sum_{m,n\in\mathbb{Z}} x_m z^m\, y_{n-m} z^{n-m} = \tau_0(x)\tau_0(y),$$

so τ_0 is a homomorphism, and

$$|\tau_0(x)| = \Big|\sum_{n\in\mathbb{Z}} x_n z^n\Big| \le \sum_{n\in\mathbb{Z}} |x_n| = \|x\|,$$

so τ_0 is continuous. Hence τ_0 extends to a continuous linear homomorphism $\tau_z \colon \ell^1(\mathbb{Z}) \to \mathbb{C}$, which is a character on $\ell^1(\mathbb{Z})$. Clearly, $\tau_z(e_1) = z$. This shows that the map $\theta \colon \mathbb{T} \to \Omega(\ell^1(\mathbb{Z}))$, $z \mapsto \tau_z$ is a bijection. We claim that θ is a homeomorphism. Indeed, θ is continuous since for $x \in \ell^1(\mathbb{Z})$ and $z \in \mathbb{T}$ we have

$$J_x(\theta(z)) = J_x(\tau_z) = \tau_z(x) = \sum_{n\in\mathbb{Z}} x_n z^n.$$

Since $x \in \ell^1(\mathbb{Z})$, this series converges absolutely (for $z \in \mathbb{T}$). The partial sums of the series are continuous functions $\mathbb{T} \to \mathbb{C}$, so $z \mapsto J_x(\theta(z))$ is continuous $\mathbb{T} \to \mathbb{C}$. By Proposition 2.2.6, θ is continuous. Since $\mathbb{T}$ is compact and $\Omega(\ell^1(\mathbb{Z}))$ is Hausdorff, Lemma 2.1.1 shows that θ is a homeomorphism.

The next lemma is purely algebraic.

3.1.10 Lemma. *Let A be a unital abelian Banach algebra and let $a \in A$. The following are equivalent:*

(i). $a \notin \operatorname{Inv} A$;

(ii). $a \in I$ for some proper ideal I of A;

(iii). $a \in M$ for some maximal ideal M of A.

Proof. (i) $\implies$ (ii): If $a \notin \operatorname{Inv} A$, consider the set $I = \{ab \colon b \in A\}$. Since A is abelian, this is an ideal of A, and since A is unital we have $a = a1 \in I$. If $1 \in I$ then $ab = 1$ for some $b \in A$, so $a \in \operatorname{Inv} A$, a contradiction. So $1 \notin I$ and I is a proper ideal.

(ii) $\implies$ (iii): Suppose that $a \in I$ where I is a proper ideal of A. By Remark 1.5.7(ii), $I \subseteq M$ for some maximal ideal M of A; so $a \in M$.

(iii) $\implies$ (i): If M is a maximal ideal of A then M is a proper ideal, so $M \cap \operatorname{Inv} A = \emptyset$ by Lemma 1.5.5. Hence $a \notin \operatorname{Inv} A$ for every $a \in M$. $\square$

3.1.11 Corollary. *Let A be a unital abelian Banach algebra and let $a \in A$.*

(i). $a \in \operatorname{Inv} A$ *if and only if* $\tau(a) \neq 0$ *for all* $\tau \in \Omega(A)$.

(ii). $\sigma(a) = \{\tau(a) \colon \tau \in \Omega(A)\}$.

(iii). $r(a) = \sup_{\tau \in \Omega(A)} |\tau(a)|$.

Proof. (i) We have:

$$\begin{aligned} a \in \operatorname{Inv} A &\iff a \notin M \text{ for all maximal ideals } M \text{ of } A, \text{ by Lemma 3.1.10} \\ &\iff a \notin \ker \tau \text{ for all } \tau \in \Omega(A), \text{ by Theorem 3.1.8} \\ &\iff \tau(a) \neq 0 \text{ for all } \tau \in \Omega(A). \end{aligned}$$

(ii) This follows from the equivalences:

$$\begin{aligned} \lambda \in \sigma(a) &\iff \lambda - a \notin \operatorname{Inv} A \\ &\iff \tau(\lambda - a) = 0 \text{ for some } \tau \in \Omega(A), \text{ by (i)} \\ &\iff \lambda = \tau(a) \text{ for some } \tau \in \Omega(A), \text{ since } \tau(\lambda) = \lambda \text{ by Lemma 3.1.4.} \end{aligned}$$

(iii) follows immediately from (ii) and the definition of $r(a)$. □

3.2 The Gelfand representation

3.2.1 Definition. Let A be a unital abelian Banach algebra. For $a \in A$, the *Gelfand transform of* a is the mapping

$$\widehat{a} \colon \Omega(A) \to \mathbb{C}, \quad \tau \mapsto \tau(a).$$

In other words, $\widehat{a} = J_a|_{\Omega(A)}$.

3.2.2 Examples. (i). Let X be a compact Hausdorff space. We have seen that the map $X \to \Omega(C(X))$, $x \mapsto \varepsilon_x$ is a homeomorphism. If $f \in C(X)$ then

$$\widehat{f} \colon \Omega(C(X)) \to \mathbb{C}, \quad \varepsilon_x \mapsto \varepsilon_x(f) = f(x).$$

This means that, if we identify $\Omega(C(X))$ with X by pretending that $x = \varepsilon_x$, then $\widehat{f} = f$.

(ii). We have seen that $\Omega(\ell^1(\mathbb{Z}))$ can be identified with $\mathbb{T}$, by pretending that $z \in \mathbb{T}$ is the same as the character $\tau_z \colon \ell^1(\mathbb{Z}) \to \mathbb{C}$, $x \mapsto \sum_{n \in \mathbb{Z}} x_n z^n$. Hence for $x \in \ell^1(\mathbb{Z})$, we have

$$\widehat{x} \colon \Omega(\ell^1(\mathbb{Z})) \to \mathbb{C}, \quad \tau_z \mapsto \tau_z(x) = \sum_{n \in \mathbb{Z}} x_n z^n.$$

If we write $z = e^{i\theta}$ and $x_n = x(n)$ then this takes the form

$$\widehat{x}(e^{i\theta}) = \sum_{n\in\mathbb{Z}} x(n)e^{in\theta},$$

so $\widehat{x}$ maybe viewed as the inverse Fourier transform of x.

3.2.3 Theorem. *Let A be a unital abelian Banach algebra. For each $a \in A$, the Gelfand transform $\widehat{a}$ is in $C(\Omega(A))$. Moreover, the mapping*

$$\gamma\colon A \to C(\Omega(A)), \quad a \mapsto \widehat{a}$$

is a unital, norm-decreasing (and hence continuous) homomorphism, and for each $a \in A$ we have

$$\sigma_A(a) = \sigma_{C(\Omega(A))}(\widehat{a}) = \{\widehat{a}(\tau) \colon \tau \in \Omega(A)\} \quad \textit{and} \quad r(a) = \|\widehat{a}\|.$$

Proof. By the definition of the topology on $\Omega(A)$, each $\widehat{a}$ is in $C(\Omega(A))$. It is easy to see that γ is a homomorphism, and it is unital by Lemma 3.1.4. The identification of $\sigma_A(a)$ with $\sigma_{C(\Omega(A))}(\widehat{a})$ follows from Corollary 3.1.11(ii) and Example 1.3.2(ii). Now $r(a) = \|\widehat{a}\| \le \|a\|$ by Corollary 3.1.11(iii) and Remark 1.3.10, so γ is linear and norm-decreasing, hence continuous. □

3.2.4 Definition. If A is a unital abelian Banach algebra then the unital homomorphism $\gamma\colon A \to C(\Omega(A))$, $a \mapsto \widehat{a}$ is called the *Gelfand representation* of A.

3.2.5 Remark. In general, the Gelfand representation is neither injective nor surjective. For example, the Gelfand representation of the disc algebra is the inclusion $A(\overline{\mathbb{D}}) \to C(\overline{\mathbb{D}})$ which is not surjective.

4 C*-algebras

4.1 Definitions and examples

4.1.1 Definition. Let A be a Banach algebra. An *involution* on A is a map $A \to A$, $a \mapsto a^*$ such that for all $a, b \in A$ and $\lambda \in \mathbb{C}$ we have:

(i). $(\lambda a)^* = \overline{\lambda} a^*$ and $(a+b)^* = a^* + b^*$ (conjugate linearity);

(ii). $(ab)^* = b^* a^*$; and

(iii). $(a^*)^* = a$.

A *C*-algebra* is a Banach algebra A equipped with an involution such that the *C*-condition* holds:

$$\|a\|^2 = \|a^* a\| \quad \text{for all } a \in A.$$

4.1.2 Remark. We usually write a^{**} instead of $(a^*)^*$.

4.1.3 Examples. (i). The zero Banach algebra $\{0\}$ is a C*-algebra.

(ii). The complex numbers $\mathbb{C}$ form a C*-algebra under the usual Banach space norm $\|\lambda\| = |\lambda|$ and the involution $\lambda^* = \overline{\lambda}$.

(iii). If X is a topological space, then $BC(X)$ is a C*-algebra under the involution $f^*(x) = \overline{f(x)}$. In particular, if X is compact, then $BC(X) = C(X)$ is a C*-algebra.

(iv). If H is a Hilbert space then $\mathcal{B}(H)$ is a C*-algebra under the involution $T \mapsto T^*$ defined by the property that $\langle Tx, y\rangle = \langle x, T^* y\rangle$ for all $x, y \in H$ (see [FA]).

(v). If A is a C*-algebra then a *C*-subalgebra* of A is a Banach subalgebra $B \subseteq A$ that is closed under the involution; in other words, B is a closed linear subspace of A such that whenever $a, b \in A$ we have $ab \in A$ and $a^* \in A$. Clearly, any C*-subalgebra of a C*-algebra is a C*-algebra.

4.1.4 Proposition. *Let A be a C*-algebra.*

(i). If A has an identity element 1 then $1^* = 1$, and if A is non-zero then $\|1\| = 1$.

(ii). If A is unital then $a \in \operatorname{Inv} A \iff a^* \in \operatorname{Inv} A$, and for $a \in \operatorname{Inv} A$ we have $(a^*)^{-1} = (a^{-1})^*$.

(iii). $\|a^*\| = \|a\|$ for all $a \in A$.

(iv). The involution $A \to A$, $a \mapsto a^*$ is continuous.

(v). $\sigma(a^*) = \sigma(a)^* = \{\overline{\lambda}\colon \lambda \in \sigma(a)\}$ *for all* $a \in A$.

Proof. (i) We have

$$1 = (1^*)^* = (1^*1)^* = 1^*1^{**} = 1^*1 = 1^*.$$

Hence $\|1\|^2 = \|1^*1\| = \|1\|$; if $A \neq \{0\}$ then $\|1\| \neq 0$ and we can cancel to obtain $\|1\| = 1$.

(ii) If $a \in \operatorname{Inv} A$ then $a^*(a^{-1})^* = (a^{-1}a)^* = 1^* = 1$ and $(a^{-1})^*a^* = (aa^{-1})^* = 1^* = 1$. Hence a^* is invertible, with inverse $(a^{-1})^*$. Conversely, if $a^* \in \operatorname{Inv} A$ then $a^{**} = a \in \operatorname{Inv} A$ by the same argument.

(iii) This is trivial for $a = 0$. If $a \neq 0$ then $\|a\|^2 = \|a^*a\| \leq \|a^*\|\,\|a\|$, which on cancelling gives $\|a\| \leq \|a^*\|$. Hence $\|a^*\| \leq \|a^{**}\| = \|a\| \leq \|a^*\|$, so we have equality.

(iv) If $a_n, a \in A$ with $a_n \to a$ then $\|a_n^* - a^*\| = \|(a_n - a)^*\| \overset{\text{(iii)}}{=} \|a_n - a\| \to 0$ as $n \to \infty$, so the involution is continuous.

(v) Without loss of generality, suppose that A is unital. For any $\lambda \in \mathbb{C}$,

$$\lambda \in \sigma(a^*) \iff \lambda 1 - a^* \overset{\text{(i)}}{=} (\overline{\lambda}1 - a)^* \notin \operatorname{Inv} A \overset{\text{(ii)}}{\iff} \overline{\lambda}1 - a \notin \operatorname{Inv} A \iff \overline{\lambda} \in \sigma(a).$$

Hence $\sigma(a^*) = \sigma(a)^*$. □

4.1.5 Definition. Let A be a C*-algebra.
An element $a \in A$ is *normal* if a commutes with a^*.
An element $a \in A$ is *hermitian* if $a = a^*$.
An element $p \in A$ is a *projection* if $p = p^* = p^2$.
If A is unital then an element $u \in A$ is *unitary* if $uu^* = u^*u = 1$ (that is, if u is invertible and $u^{-1} = u^*$).

4.1.6 Remark. Clearly, projections are hermitian, and both unitary and hermitian elements are normal.

4.1.7 Proposition. *Let A be a C*-algebra.*

(i). If $a \in A$ then a^*a is hermitian.

(ii). If $p \in A$ is a non-zero projection then $\|p\| = 1$.

(iii). Every $a \in A$ may be written uniquely in the form $a = h + ik$ where h and k are hermitian elements of A, called the real *and* imaginary *parts of a.*

Proof. (i) If $h = a^*a$ then $h^* = (a^*a)^* = a^*a^{**} = a^*a = h$, so h is hermitian.

(ii) We have $\|p\|^2 = \|p^*p\| = \|p^2\| = \|p\|$. Since $\|p\| \neq 0$ we may cancel to obtain $\|p\| = 1$.

(iii) We have $a = h + ik$ where $h = \frac{1}{2}(a + a^*)$ and $k = \frac{1}{2i}(a - a^*)$, as is easily verified. If $h + ik = h' + ik'$ where h, h', k, k' are hermitian, then $i(k' - k) = h - h' = (h - h')^* = \big(i(k' - k)\big)^* = -i(k' - k)$, hence $h = h'$ and $k = k'$, demonstrating uniqueness. □

4.1.8 Lemma. *If A is a unital C*-algebra and $\tau \in A^*$ with $\|\tau\| \leq 1$ and $\tau(1) = 1$, then $\tau(h) \in \mathbb{R}$ for all hermitian elements $h \in A$.*

Proof. Suppose that $\tau(h) = x + iy$ where $x, y \in \mathbb{R}$. Observe that for $t \in \mathbb{R}$ we have $\tau(h + it1) = \tau(h) + it = x + i(y + t)$, so

$$x^2+(y+t)^2 = |\tau(h+it)|^2 \leq \|h+it\|^2 = \|(h-it)(h+it)\| = \|h^2+t^2\| \leq \|h^2\|+t^2$$

and so $x^2 + 2yt \leq \|h^2\|$ for all $t \in \mathbb{R}$. This forces $y = 0$, so $\tau(h) = x \in \mathbb{R}$. □

4.1.9 Proposition. *If A is a unital abelian C*-algebra then $\tau(a^*) = \overline{\tau(a)}$ for all $a \in A$ and $\tau \in \Omega(A)$.*

Proof. If $\tau \in \Omega(A)$ then $\|\tau\| = \tau(1) = 1$ by Lemma 3.1.4. If $a \in A$ then $a = h + ik$ for some hermitian $h, k \in A$ by Proposition 4.1.7(iii), so $\tau(h)$ and $\tau(k)$ are real by Lemma 4.1.8. Hence

$$\tau(a^*) = \tau(h - ik) = \tau(h) - i\tau(k) = \overline{\tau(h) + i\tau(k)} = \overline{\tau(a)}.$$ □

4.1.10 Corollary. *Let A be a unital C*-algebra.*

(i). If $h \in A$ is hermitian then $\sigma_A(h) \subseteq \mathbb{R}$.

(ii). If $u \in A$ is unitary then $\sigma_A(u) \subseteq \mathbb{T}$.

Proof. If $a \in A$ and a commutes with a^* then let $C = \{a, a^*\}''$. This is a closed unital abelian subalgebra of A and $\sigma_A(a) = \sigma_C(a) = \{\tau(a) \colon \tau \in \Omega(C)\}$ by Proposition 1.3.19 and Theorem 3.2.3. The commutant of a self-adjoint subset of A is self-adjoint (that is, it is closed under the involution), so C is self-adjoint. Hence C is a unital abelian C*-subalgebra of A.

(i) If $a = h$ is hermitian then $\tau(a) \in \mathbb{R}$ for all $\tau \in \Omega(C)$ by Proposition 4.1.9, which establishes the result.

(ii) If $a = u$ is unitary then by Proposition 4.1.9 we have

$$|\tau(u)|^2 = \tau(u)^*\tau(u) = \tau(u^*)\tau(u) = \tau(u^*u) = \tau(1) = 1$$

for every $\tau \in \Omega(C)$, so $\sigma_A(u) \subseteq \mathbb{T}$. □

4.1.11 Corollary. *Let A be a unital C*-algebra and let $B \subseteq A$ be a C*-subalgebra with $1 \in B$. If $b \in B$ then $\sigma_B(b) = \sigma_A(b)$.*

Proof. If b is hermitian then $\sigma_A(b) \subseteq \mathbb{R}$ by Corollary 4.1.10(i). Therefore $\sigma_A(b)$ has no holes, so $\sigma_B(b) = \sigma_A(b)$ by Theorem 1.3.16.

If b is any element of $B \cap \operatorname{Inv} A$ and if a is the inverse of b in A then b^*b is invertible in A with inverse aa^*. Since b^*b is hermitian, the first paragraph shows that b^*b is invertible in B; so $aa^* \in B$. Hence $aa^*b^* \in B$ and $baa^*b^* = aa^*b^*b = 1$, so b is invertible in B with inverse aa^*b^*. Hence $b \in \operatorname{Inv} B$.

This shows that $\operatorname{Inv} B = B \cap \operatorname{Inv} A$, so $\sigma_B(b) = \sigma_A(b)$ for any $b \in B$. □

4.1.12 Proposition.
If a is a hermitian element of a C-algebra then $\|a\| = r(a)$.*

Proof. Since $a = a^*$ we have $\|a\|^2 = \|a^*a\| = \|a^2\|$, so for $n \geq 1$ we have $\|a\|^{2^n} = \|a^{2^n}\|$ by induction. Hence, by Theorem 1.3.12,

$$r(a) = \lim_{n\to\infty} \|a^{2^n}\|^{1/2^n} = \|a\|. \qquad \square$$

4.1.13 Definition. If A and B are C*-algebras then a $*$-*homomorphism* $\theta\colon A \to B$ is a homomorphism of Banach algebras which respects the involution: $\theta(a^*) = \theta(a)^*$ for all $a \in A$. If A and B are unital and $\theta(1) = 1$ then we say that θ is *unital*. An invertible $*$-homomorphism is called a $*$-*isomorphism*.

4.1.14 Proposition. *Let A and B be unital C*-algebras and let $\theta\colon A \to B$ be a unital $*$-homomorphism.*

(i). $\theta(\operatorname{Inv} A) \subseteq \operatorname{Inv} B$, and $\theta(a)^{-1} = \theta(a^{-1})$ for $a \in \operatorname{Inv} A$.

(ii). For all $a \in A$ we have $\sigma_A(a) \supseteq \sigma_B(\theta(a))$.

(iii). θ is continuous; in fact, $\|\theta(a)\| \leq \|a\|$ for all $a \in A$.

(iv). If θ is a $*$-isomorphism then θ is an isometry and $\sigma_A(a) = \sigma_B(\theta(a))$ for all $a \in A$.

Proof. (i) and (ii) are special cases of Proposition 1.5.12.

(iii) Let $a \in A$. We have

$$\begin{aligned}
\|\theta(a)\|^2 = \|\theta(a)^*\theta(a)\| &= r_B(\theta(a)^*\theta(a)) \text{ by Proposition 4.1.12} \\
&= r_B(\theta(a^*a)) \\
&\leq r_A(a^*a) \text{ by (ii)} \\
&= \|a^*a\| \text{ by Proposition 4.1.12} \\
&= \|a\|^2.
\end{aligned}$$

So $\|\theta(a)\| \leq \|a\|$ for all $a \in A$. Since θ is a linear map, this shows that θ is continuous (with $\|\theta\| \leq 1$).

(iv) If θ is a $*$-isomorphism then θ is a bijective $*$-homomorphism, from which it is easy to see that θ^{-1} is also a $*$-homomorphism. By (iii) both θ and θ^{-1} are continuous linear maps with norm no greater than 1, so

$$\|a\| = \|\theta^{-1}(\theta(a))\| \leq \|\theta(a)\| \leq \|a\|$$

for each $a \in A$. Hence we have equality throughout and θ is an isometry. We showed that $\sigma_A(a) = \sigma_B(\theta(a))$ for all $a \in A$ in Proposition 1.5.12. $\square$

4.2 The Stone-Weierstrass theorem

4.2.1 Lemma. *If $m \in \mathbb{N}$ then there is a sequence of polynomials $p_1, p_2, p_3, \ldots$ with real coefficients such that $0 \leq p_n(t) \leq 1$ for $n \geq 1$ and $0 \leq t \leq 1$, and*

$$p_n \to 0 \text{ uniformly on } [0, \tfrac{1}{2m}] \quad \text{and} \quad p_n \to 1 \text{ uniformly on } [\tfrac{2}{m}, 1].$$

Proof. A calculus exercise shows that if $N \in \mathbb{N}$ then

$$1 - (1-x)^N \leq Nx \quad \text{and} \quad (1-x)^N \leq \frac{1}{Nx} \quad \text{for each } x \in (0,1).$$

Let $p_n(t) = 1 - (1-t^n)^{m^n}$. We have $0 \leq p_n(t) \leq 1$ for $t \in [0,1]$, and

$$\sup_{t\in[0,1/2m]} |p_n(t)| = \sup_{t\in[0,1/2m]} 1 - (1-t^n)^{m^n} \leq \sup_{t\in[0,1/2m]} (mt)^n \leq 2^{-n} \to 0 \text{ as } n \to \infty,$$

and similarly,

$$\sup_{t\in[2/m,1]} |1-p_n(t)| = \sup_{t\in[2/m,1]} (1-t^n)^{m^n} \leq \sup_{t\in[2/m,1]} \frac{1}{(mt)^n} \leq 2^{-n} \to 0 \text{ as } n \to \infty.$$

$\square$

Let X be a compact Hausdorff space, and let $C(X,\mathbb{R})$ denote the set of continuous functions $X \to \mathbb{R}$. This is a real Banach space under the uniform norm [FA 1.7.2], and it is easy to check that the pointwise product turns it into a real Banach algebra. The basic definitions we made for complex Banach algebras carry over simply by changing $\mathbb{C}$ to $\mathbb{R}$, so we may talk of subalgebras and unital subalgebras of $C(X,\mathbb{R})$.

If $A \subseteq C(X,\mathbb{R})$ (or $A \subseteq C(X)$), we say that A *separates the points of* X if for every pair of distinct points $x, y \in X$, there is some $f \in A$ such that $f(x) \neq f(y)$.

4.2.2 Theorem (The real-valued Stone-Weierstrass theorem). *Let X be a compact Hausdorff topological space and let $A \subseteq C(X, \mathbb{R})$. If A is a unital subalgebra of $C(X, \mathbb{R})$ which separates the points of X, then A is uniformly dense in $C(X, \mathbb{R})$.*

Proof. We will write statements such as "$f \leq 1$" to mean $f(x) \leq 1$ for all $x \in X$, and "$f \leq 1$ on K" to mean that $f(x) \leq 1$ for all $x \in K$. We split the proof into several steps.

(i). If $x, y \in X$ with $x \neq y$ then there is a function $f \in A$ with $f \geq 0$ such that $f(x) = 0$ and $f(y) > 0$.

Since A separates the points of X, there is a function $g \in A$ with $g(x) \neq g(y)$. Let $f = (g - g(x)1)^2$. Since A is a unital algebra, $f \in A$, and clearly $f \geq 0$. Moreover, $f(x) = 0$ and $f(y) > 0$.

(ii). If L is a compact subset of X and $x \in X \setminus L$ then there is some $f \in A$ with $0 \leq f \leq 1$ such that $f(x) = 0$ and $f(y) > 0$ for all $y \in L$.

By (i), for each $y \in L$ there is a function $f_y \in A$ such that $f_y \geq 0$, $f_y(x) = 0$ and $f_y(y) > 0$. Since f_y is continuous, there is an open set U_y containing y such that $f_y > 0$ on U_y. As y varies over L, the sets U_y cover L. Since L is compact, there are $y_1, \dots, y_k \in L$ such that $U_{y_1}, \dots, U_{y_k}$ cover L. Let $f_0 = f_{y_1} + \dots + f_{y_k}$ and let $f = f_0 / \|f_0\|$. Then $f \in A$, $0 \leq f \leq 1$, $f(x) = 0$ and $f(y) > 0$ for $y \in L$.

(iii). If K, L are disjoint compact subsets of X then there is a function $f \in A$ with $0 \leq f \leq 1$ such that $f \leq \frac{1}{4}$ on K and $f \geq \frac{3}{4}$ on L.

Let $x \in K$. By (ii), there is a function $g_x \in A$ such that $0 \leq g_x \leq 1$, $g_x(x) = 0$, and $g_x > 0$ on L. Since L is compact, $g_x(L)$ is compact, so is contained in a set of the form $[s, t]$ for some $s > 0$. Hence there is $m \in \mathbb{N}$ (depending on x) such that $g_x \geq \frac{2}{m}$ on L. For $x \in K$, let $U_x = \{z \in X : g_x(z) < \frac{1}{2m}\}$. Since $x \in U_x$, these open sets cover K, so by compactness there are $x_1, \dots, x_k \in K$ such that $U_{x_1}, \dots, U_{x_k}$ cover K.

By Lemma 4.2.1, for $i = 1, \dots, k$ there is a polynomial p (depending on i) such that if we write $f_i = p(g_{x_i})$, then $0 \leq f_i \leq 1$, $f_i \leq \frac{1}{4}$ on U_{x_i} and $f_i \geq (\frac{3}{4})^{1/k}$ on L. Since p is a polynomial and $g_{x_i} \in A$, we have $f_i \in A$.

Now let $f = f_1 f_2 \dots f_k$. Then $f \in A$ and f has the desired properties.

(iv). If $f \in C(X, \mathbb{R})$ then there is $g \in A$ such that $\|f - g\| \leq \frac{3}{4}\|f\|$.

We may assume (by considering $f/\|f\|$) that $\|f\| = 1$. Let $h \in A$ be the function obtained by applying (iii) to $K = \{x \in X : f(x) \leq -\frac{1}{4}\}$

and $L = \{x \in X\colon f(x) \geq \frac{1}{4}\}$, and let $g = h - \frac{1}{2}$. Since $\frac{1}{4} \leq g \leq \frac{1}{2}$ on L, we have $|f - g| \leq \frac{3}{4}$ on L; similarly, $|f - g| \leq \frac{3}{4}$ on K. Since $-\frac{1}{2} \leq g \leq \frac{1}{2}$ and $-\frac{1}{4} \leq f \leq \frac{1}{4}$ on $X \setminus (K \cup L)$ we have $|f - g| \leq \frac{3}{4}$ on $X \setminus (K \cup L)$. Hence $\|f - g\| \leq \frac{3}{4}$.

(v). *A is uniformly dense in $C(X, \mathbb{R})$.*

Let $f \in C(X, \mathbb{R})$. By (iv), there is $g_1 \in A$ with $\|f - g_1\| \leq \frac{3}{4}\|f\|$. Applying (iv) to the function $f - g_1$, we obtain a function $g_2 \in A$ with $\|f - g_1 - g_2\| \leq \frac{3}{4}\|f - g_1\| \leq (\frac{3}{4})^2\|f\|$. Continuing in this manner, we obtain functions $g_1, g_2, g_3, \ldots \in A$ such that

$$\|f - (g_1 + \cdots + g_n)\| \leq (\tfrac{3}{4})^n\|f\| \to 0 \quad \text{as } n \to \infty.$$

Since $g_1 + \cdots + g_n \in A$, this shows that f is in the uniform closure of A. $\square$

4.2.3 Theorem (The complex Stone-Weierstrass theorem)**.** *Let X be a compact Hausdorff topological space and let $A \subseteq C(X)$. If A is a unital $*$-subalgebra of $C(X)$ which separates the points of X, then A is uniformly dense in $C(X)$.*

Proof. For $f \in C(X)$, let $f_1(x) = \mathrm{Re}(f(x))$ and $f_2(x) = \mathrm{Im}(f(x))$. Then $f_1, f_2 \in C(X, \mathbb{R})$. Moreover, $f_1 = \frac{1}{2}(f + f^*)$ and $f_2 = \frac{1}{2i}(f - f^*)$ are in A since A is a complex vector space which is closed under the $*$-operation.

Consider the set $A_1 = \{f_1\colon f \in A\}$. Since A separates the points of X, if $x, y \in X$ with $x \neq y$ then there is $f \in A$ such that $f(x) \neq f(y)$. Hence either $f_1(x) \neq f_1(y)$ or $f_2(x) \neq f_2(y)$. If $f_1(x) \neq f_1(y)$ then f_1 is a function in A_1 separating x and y, and if $f_2(x) \neq f_2(y)$ then $g = -if = f_2 - if_1 \in A$ and $f_2 = g_1$ is a function in A_1 separating x and y.

This shows that A_1 separates the points of X, and it is easy to see that A_1 is a unital subalgebra of $C(X, \mathbb{R})$. By the real-valued Stone-Weierstrass theorem, A_1 is uniformly dense in $C(X, \mathbb{R})$.

If $f \in C(X)$ then $f = f_1 + if_2$. Given $\varepsilon > 0$ there is a function $g_1 \in A_1$ with $\|f_1 - g_1\|_\infty < \varepsilon/2$ and a function $g_2 \in A_1$ with $\|f_2 - g_2\|_\infty < \varepsilon/2$. Hence $g = g_1 + ig_2 \in A$ and $\|f - g\|_\infty \leq \|f_1 - g_1\|_\infty + \|i(f_2 - g_2)\|_\infty < \varepsilon$, so A is uniformly dense in $C(X)$. $\square$

4.3 Abelian C*-algebras and the continuous functional calculus

We now apply Gelfand's theory of abelian unital Banach algebras (§3) to show that, up to isometric $*$-isomorphism, every unital abelian C*-algebra is of the form $C(X)$ for some compact Hausdorff space X.

4.3.1 Theorem (The Gelfand-Naimark theorem)**.** *If A is a unital abelian C^*-algebra then the Gelfand representation of A,*

$$\gamma\colon A \to C(\Omega(A)), \quad a \mapsto \widehat{a}$$

is a unital isometric $*$-isomorphism of A onto $C(\Omega(A))$.

Proof. Theorem 3.2.3 tells us that γ is a unital, norm-decreasing homomorphism with $\|\gamma(a)\| = r(a)$. Moreover, for $a \in A$ and $\tau \in \Omega(A)$ we have

$$\widehat{a^*}\,(\tau) = \tau(a^*) = \overline{\tau(a)} = \overline{\widehat{a}(\tau)} = (\widehat{a})^*(\tau)$$

by Proposition 4.1.9, so $\gamma(a^*) = \gamma(a)^*$ and γ is a $*$-homomorphism. Now a^*a is hermitian by Proposition 4.1.7(i), so by Proposition 4.1.12,

$$\|\gamma(a)\|^2 = \|\gamma(a)^*\gamma(a)\| = \|\gamma(a^*a)\| = r(a^*a) = \|a^*a\| = \|a\|^2.$$

Hence γ is an isometry.

It remains to show that γ is surjective, for which we appeal to the Stone-Weierstrass theorem. Recall that $\Omega(A)$ is a compact Hausdorff space by Theorem 3.1.6. Since γ is a unital $*$-homomorphism, its image $\gamma(A)$ is a unital $*$-subalgebra of $C(\Omega(A))$. If $\tau_1, \tau_2 \in \Omega(A)$ with $\tau_1 \neq \tau_2$ then there is $a \in A$ with $\tau_1(a) \neq \tau_2(a)$, hence $\widehat{a}(\tau_1) \neq \widehat{a}(\tau_2)$ and so $\gamma(A)$ separates the points of $\Omega(A)$. By the Stone-Weierstrass theorem 4.2.3, $\gamma(A)$ is dense in $C(\Omega(A))$. Since γ is a linear isometry its range is closed. Hence

$$\gamma(A) = \overline{\gamma(A)} = C(\Omega(A)). \qquad \square$$

4.3.2 Corollary. *If A is a unital abelian C^*-algebra and $a, b \in A$ with $\tau(a) = \tau(b)$ for all $\tau \in \Omega(A)$, then $a = b$.*

Proof. $\widehat{a} = \widehat{b}$, so $a = b$ by the Gelfand-Naimark theorem. $\square$

4.3.3 Definition. If S is a subset of a C*-algebra A, then we write $C^*(S)$ for the smallest C*-subalgebra of A containing S. In particular, if A is a unital C*-algebra and $a \in A$ then we will write $C^*(1, a) = C^*(\{1, a\})$.

4.3.4 Lemma. *Let A be a C^*-algebra and let $S \subseteq A$.*

(i). The linear span of the elements of the form $a_1^{n_1}\, a_2^{n_2}\, \dots\, a_k^{n_k}$ for $k \geq 1$, $n_1, \dots, n_k \geq 1$ and $a_i \in S$ or $a_i^* \in S$ for $1 \leq i \leq k$ is dense in $C^*(S)$.

(ii). Suppose that A is unital and that B is another unital C^*-algebra. If $A = C^*(S)$ and θ_1, θ_2 are two unital $*$-homomorphisms $A \to B$ such that $\theta_1(a) = \theta_2(a)$ for all $a \in S$, then $\theta_1 = \theta_2$.

Proof. (i) Clearly, any C*-algebra containing S must also contain every element of the given form, hence $C^*(S)$ contains the closed linear span of such elements. Conversely, it is easy to see that the closed linear span of these elements forms a C*-algebra containing S, so this C*-algebra is equal to $C^*(S)$.

(ii) Since θ_1 and θ_2 are unital $*$-homomorphisms, they are continuous by Proposition 4.1.14(iii), and we have $\theta_1(b) = \theta_2(b)$ for any $b = a_1^{n_1} a_2^{n_2} \dots a_n^{n_k}$ with $a_i \in S$ or $a_i^* \in S$ for $1 \le i \le k$. Since θ_1 and θ_2 are linear, they agree on the linear span of such elements, which is dense by (i). Continuous maps agreeing on a dense subset of their domain are equal, so $\theta_1 = \theta_2$. □

4.3.5 Lemma. *Let A be a unital C*-algebra. If a is a normal element of A then $C^*(1, a)$ is a unital abelian C*-algebra.*

Proof. Exercise. □

We can now prove a generalisation of Proposition 4.1.12.

4.3.6 Corollary. *If a is a normal element of a unital C*-algebra A then $r(a) = \|a\|$.*

Proof. Let $B = C^*(1, a)$. By Corollary 1.3.13, $r(a) = r_A(a) = r_B(a)$. By the Gelfand-Naimark theorem 4.3.1, if $\gamma\colon B \to C(\Omega(B))$ is the Gelfand representation of B then $\|a\| = \|\gamma(a)\| = r_B(a) = r(a)$. □

4.3.7 Lemma. *Suppose that a is a normal element of a unital C*-algebra, let $\Omega = \Omega(C^*(1, a))$ and let $\gamma\colon C^*(1, a) \to C(\Omega)$ be the Gelfand representation of $C^*(1, a)$. The map $\widehat{a} = \gamma(a)\colon \Omega \to \mathbb{C}$ is a homeomorphism of Ω onto $\sigma(a)$.*

Proof. By Lemma 3.1.4 and Proposition 4.1.9, every $\tau \in \Omega$ is a unital $*$-homomorphism $C^*(1, a) \to \mathbb{C}$. If $\widehat{a}(\tau_1) = \widehat{a}(\tau_2)$ for some $\tau_1, \tau_2 \in \Omega$ then $\tau_1(a) = \tau_2(a)$ and $\tau_1(1) = 1 = \tau_2(1)$. Taking $S = \{1, a\}$ in Lemma 4.3.4(ii) we see that $\tau_1 = \tau_2$. Hence $\widehat{a}$ is injective.

The map $\widehat{a}$ is continuous by the definition of the topology on Ω. Moreover, Ω is compact by Theorem 3.1.6 and $\widehat{a}(\Omega) = \sigma(a)$ is Hausdorff. By Lemma 2.1.1, $\widehat{a}$ is a homeomorphism onto $\sigma(a)$. □

4.3.8 Lemma. *Suppose that X and Y are compact Hausdorff topological spaces and $\psi\colon X \to Y$ is a homeomorphism. The map $\psi^t\colon C(Y) \to C(X)$, $f \mapsto f \circ \psi$ is an isometric unital $*$-isomorphism.*

Proof. Exercise. □

4.3.9 Theorem (The continuous functional calculus)**.** *Let a be a normal element of a unital C*-algebra A. There is a unique unital $*$-homomorphism $\theta_a\colon C(\sigma(a)) \to A$ such that $\theta_a(z) = a$ where $z \in C(\sigma(a))$ is the function $z(\lambda) = \lambda$ for $\lambda \in \sigma(a)$. Moreover, θ_a is an isometric $*$-isomorphism onto $C^*(1, a)$.*

Proof. Suppose that θ_1 and θ_2 are two unital $*$-homomorphisms $C(\sigma(a)) \to A$ with $\theta_1(z) = \theta_2(z) = a$. Since $z(\lambda) = z(\mu)$ if and only if $\lambda = \mu$, the function z separates the points of $\sigma(a)$, so $C^*(\{1, z\})$ is a closed separating unital $*$-subalgebra of $C(\sigma(a))$. By the Stone-Weierstrass theorem 4.2.3, $C^*(\{1, z\}) = C(\sigma(a))$, so $\theta_1 = \theta_2$ by Lemma 4.3.4(ii). This proves the uniqueness statement.

Let $\Omega = \Omega(C^*(1, a))$, let $\gamma\colon C^*(1, a) \to C(\Omega)$ be the Gelfand representation of $C^*(1, a)$ and let $\widehat{a} = \gamma(a)$. By the Gelfand-Naimark theorem 4.3.1 and the last two lemmas, γ and $(\widehat{a})^t\colon C(\sigma(a)) \to C(\Omega)$ are both isometric unital $*$-isomorphisms. Consider the map $\theta_a\colon C(\sigma(a)) \to C^*(1, a)$ given by $\theta_a = \gamma^{-1} \circ (\widehat{a})^t$:

$$\begin{tikzcd} C(\sigma(a)) \arrow[rr, "\theta_a"] \arrow[dr, "(\widehat{a})^t"'] & & C^*(1,a) \\ & C(\Omega) \arrow[ur, "\gamma^{-1}"'] & \end{tikzcd}$$

Since it is the composition of two isometric $*$-isomorphisms, θ_a is an isometric $*$-isomorphism onto $C^*(1, a)$. $\square$

4.3.10 Definition. If a is a normal element of a unital C*-algebra A and $f \in C(\sigma(a))$ then we write $f(a)$ for the element $\theta_a(f) \in A$.

4.3.11 Remark. If $p = \lambda_0 + \lambda_1 z + \cdots + \lambda_n z^n$ is a complex polynomial on $\sigma(a)$, then $\theta_a(p)$ is equal to the element $p(a)$ defined in Definition 1.3.7, because θ_a is a unital homomorphism with $\theta_a(z) = a$. So this new notation is consistent.

4.3.12 Remark. This notation is quite convenient. By way of example, we use it to restate the fact that θ_a is a $*$-homomorphism. For any $f, g \in C(\sigma(a))$ and $\mu \in \mathbb{C}$, we have

$$(f + g)(a) = f(a) + g(a), \quad (\mu f)(a) = \mu\, f(a),$$
$$(fg)(a) = f(a)g(a) \quad \text{and} \quad f^*(a) = f(a)^*$$

where the functions $f + g, \mu f, fg$ and f^* are given by applying the usual pointwise operations in the C*-algebra $C(\sigma(a))$.

4.3.13 Theorem (Spectral mapping for the continuous functional calculus)**.** *If a is a normal element of a unital C*-algebra A and $f \in C(\sigma(a))$ then*

$$\sigma(f(a)) = f(\sigma(a)) = \{f(\lambda) \colon \lambda \in \sigma(a)\}.$$

Proof. Since θ_a is a unital $*$-isomorphism of $C(\sigma(a))$ onto $C^*(1,a)$, we have

$$\begin{aligned}\sigma_A(f(a)) &= \sigma_{C^*(1,a)}(f(a)) && \text{by Corollary 4.1.11}\\ &= \sigma_{C^*(1,a)}(\theta_a(f)) && \text{by the definition of } f(a)\\ &= \sigma_{C(\sigma(a))}(f) && \text{by Proposition 4.1.14(iv)}\\ &= f(\sigma(a)) && \text{by Example 1.3.2(ii).}\end{aligned}$$

□

4.4 Positive elements of C*-algebras

Let us write $\mathbb{R}^+$ for the non-negative real numbers.

4.4.1 Definition. Let A be a C*-algebra. If $a \in A$ then we say that a is *positive* and write $a \geq 0$ if a is hermitian with $\sigma(a) \subseteq \mathbb{R}^+$. The set of positive elements of A will be denoted by $A^+ = \{a \in A \colon a \geq 0\}$.

4.4.2 Example. In the case $A = C(X)$ where X is a compact Hausdorff space, we have

$$\begin{aligned}C(X)^+ &= \{f \in C(X) \colon f(x) \in \mathbb{R}^+ \text{ for } x \in X\}\\ &= \{f \in C(X) \colon f = f^* \text{ and } \|f - t\| \leq t \text{ for some } t \in \mathbb{R}^+\}\\ &= \{g^*g \colon g \in C(X)\}.\end{aligned}$$

4.4.3 Corollary. *Let A be a unital C*-algebra. For every $a \in A^+$ there is a unique element $b \in A^+$ with $b^2 = a$.*

Proof. Since $\sigma(a) \subseteq \mathbb{R}^+$, the function $f \colon \sigma(a) \to \mathbb{C}$, $t \mapsto \sqrt{t}$ is well-defined, continuous and takes values in $\mathbb{R}^+$. Hence we can define $b = f(a)$, and since $f = f^*$ we have $b^* = f^*(a) = f(a) = b$ so b is hermitian. By Theorem 4.3.13, $\sigma(b) = f(\sigma(a)) \subseteq \mathbb{R}^+$, so $b \in A^+$. Since $f^2 = z$ is the identity function on $\sigma(a)$, by Theorem 4.3.9 we have $b^2 = f(a)^2 = (f^2)(a) = z(a) = a$. This establishes the existence of $b \in A^+$ with $b^2 = a$.

To see that b is the unique element with these properties, suppose that $c \in A^+$ with $c^2 = a$. Then $ac = c^3 = ca$, so $B = C^*(\{1, a, c\})$ is a unital abelian C*-algebra and $b = f(a) \in C^*(\{1, a\}) \subseteq B$. If $\tau \in \Omega(B)$ then

$$\tau(b)^2 = \tau(b^2) = \tau(a) = \tau(c^2) = \tau(c)^2;$$

since $b, c \in A^+$ we have $\tau(b), \tau(c) \geq 0$, so $\tau(b) = \tau(c)$. By Corollary 4.3.2, $b = c$. □

4.4.4 Definition. If A is a unital C*-algebra and $a \in A^+$ then we call the element $b \in A^+$ with $b^2 = a$ the *positive square root of* a, and write it as $a^{1/2}$.

4.4.5 Lemma. *Let A be a unital C*-algebra.*

(i). If a is a hermitian element of A then $a^2 \geq 0$.

(ii). If a is a hermitian element of A then $a \geq 0$ if and only if $\|a - t\| \leq t$ for some $t \in \mathbb{R}^+$.

(iii). If $a, b \in A$ with $a \geq 0$ and $b \geq 0$ then $a + b \geq 0$.

(iv). If $a \in A$ with $a \geq 0$ and $-a \geq 0$ then $a = 0$.

(v). If $a \in A$ and $-a^*a \geq 0$ then $a = 0$.

Proof. (i) Since $\sigma(a) \subseteq \mathbb{R}$ by Corollary 4.1.10(i), the spectral mapping theorem 1.3.8 gives $\sigma(a^2) = \{\lambda^2 \colon \lambda \in \sigma(a)\} \subseteq \mathbb{R}^+$, so $a^2 \geq 0$.

(ii) If $a \geq 0$ then $\sigma(a) \subseteq [0, t]$ where $t = r(a) \in \mathbb{R}^+$. By Theorem 1.3.8 we have $\sigma(a - t) = \sigma(a) - t \subseteq [-t, 0]$ and $a - t$ is hermitian, so we have $\|a - t\| = r(a - t) \leq t$ by Proposition 4.1.12.

Conversely, if $a = a^*$ and $\|a - t\| \leq t$ for some $t \in \mathbb{R}^+$ then $\sigma(a - t) \subseteq [-t, t]$, so $\sigma(a) = \sigma(a - t) + t \subseteq [0, 2t]$ by Theorem 1.3.8. Hence $a \geq 0$.

(iii) Clearly, $a + b$ is hermitian, and by (ii) there exist $s, t \in \mathbb{R}^+$ such that $\|a - s\| \leq s$ and $\|b - t\| \leq t$. By the triangle inequality,

$$\|a + b - (s + t)\| \leq \|a - s\| + \|b - t\| \leq s + t,$$

so $a + b \geq 0$ by (ii).

(iv) We have $\sigma(a) \subseteq \mathbb{R}^+$ and $\sigma(-a) = -\sigma(a) \subseteq \mathbb{R}^+$, so $\sigma(a) = \{0\}$. Hence $\|a\| = r(a) = 0$ by Proposition 4.1.12, so $a = 0$.

(v) Using Proposition 4.1.7(iii), write $a = h + ik$ where h and k are hermitian elements of A. By Proposition 1.3.3(ii), $-aa^*$ is also positive. Now $a^*a + aa^* = 2(h^2 + k^2)$, so $a^*a = 2(h^2 + k^2) - aa^* \geq 0$ by (i) and (iii). By (iv), $a^*a = 0$, so $\|a\|^2 = \|a^*a\| = 0$ and $a = 0$. □

4.4.6 Theorem. *If A is a unital C*-algebra then $A^+ = \{a^*a \colon a \in A\}$.*

Proof. If $b \in A^+$ then $b = a^2 = a^*a$ where $a = b^{1/2}$, so $A^+ \subseteq \{a^*a \colon a \in A\}$.

It remains to show that if $a \in A$ then $b = a^*a \geq 0$. By Proposition 4.1.7(i), b is hermitian. Consider the three functions $z, z^+, z^- \in C(\sigma(b))$ given by

$$z(\lambda) = \lambda, \quad z^+(\lambda) = \begin{cases} \lambda & \text{if } \lambda \geq 0 \\ 0 & \text{if } \lambda < 0 \end{cases} \quad \text{and} \quad z^-(\lambda) = \begin{cases} 0 & \text{if } \lambda \geq 0 \\ -\lambda & \text{if } \lambda < 0 \end{cases}$$

Observe that $z^{\pm}(\sigma(b)) \subseteq \mathbb{R}^+$, $z = z^+ - z^-$ and $z^+z^- = 0$. Thus, writing $b^+ = z^+(b)$ and $b^- = z^-(b)$, we have $b^{\pm} \geq 0$ by Theorem 4.3.13, and $b = b^+ - b^-$ and $b^+b^- = 0$ by Theorem 4.3.9.

Let $c = ab^-$. We have

$$-c^*c = -b^-a^*ab^- = -b^-(b^+ - b^-)b^- = (b^-)^3 \geq 0.$$

By Lemma 4.4.5(v), $c = 0$, so $(b^-)^3 = 0$. Hence $\sigma(b^-) = \{0\}$; since b^- is hermitian, Proposition 4.1.12 shows that $b^- = 0$, so $b = b^+ \geq 0$. □

4.4.7 Definition. If $a, b \in A$ then let us write $a \leq b$ if $b - a \geq 0$.

4.4.8 Remark. It follows from parts (iii) and (iv) of Lemma 4.4.5 that the relation $\leq$ is a partial order on A.

4.4.9 Lemma. *If A is a unital C^*-algebra and $a \in A^+$ then $a \leq \|a\|1$.*

Proof. Exercise. □

4.4.10 Corollary. *Let A be a unital C^*-algebra.*

(i). If $a, b \in A$ with $a \leq b$ then $c^*ac \leq c^*bc$ for any $c \in A$.

(ii). If $a, b \in A$ then $b^*a^*ab \leq \|a\|b^*b$.

Proof. (i) Let $d = (b-a)^{1/2}$. Then $c^*bc - c^*ac = c^*d^2c = (dc)^*(dc) \geq 0$ by Theorem 4.4.6.

(ii) We have $a^*a \leq \|a\|^2 1$ by Lemma 4.4.9, so $b^*a^*ab \leq b^*(\|a\|^2 1)b = \|a\|^2 b^*b$ by (i). □

4.5 The GNS representation

If V is a vector space, recall that a positive semi-definite sesquilinear form on V is a mapping $(\cdot,\cdot)\colon V \times V \to \mathbb{C}$ such that

$$(x,x) \geq 0, \quad (\lambda x + \mu y, z) = \lambda(x,z) + \mu(y,z) \quad \text{and} \quad (y,x) = \overline{(x,y)}$$

for all $x, y, z \in V$ and $\lambda, \mu \in \mathbb{C}$. If this mapping also satisfies

$$(x,x) = 0 \implies x = 0$$

then we say that it is a positive definite, and it is then an inner product on V (see [FA 4.1]).

You will probably have seen the next result proven for inner products, but perhaps not for positive semi-definite sesquilinear forms. The proof is basically the same, though.

4.5.1 Lemma (The Cauchy-Schwarz inequality)**.** *Let V be a vector space and let $(\cdot,\cdot)\colon V\times V\to\mathbb{C}$ be a positive semi-definite sesquilinear form. Then*

$$|(x,y)|^2 \le (x,x)\,(y,y) \quad \text{for all } x,y\in V.$$

Proof. For $\lambda\in\mathbb{C}$ we have

$$0\le(\lambda x-y,\lambda x-y)=|\lambda|^2(x,x)-\lambda(x,y)-\overline{\lambda}(y,x)+(y,y).$$

Taking $\lambda=t\overline{(x,y)}$ for $t\in\mathbb{R}$ gives

$$p(t)=t^2|(x,y)|^2(x,x)-2t|(x,y)|^2+(y,y)\ge 0 \quad \text{for all } t\in\mathbb{R}.$$

Observe that $p(t)$ is a polynomial in t. If $p(t)$ is constant then $(x,y)=0$ and we are done. Otherwise, $|(x,y)|^2\ne 0$ and since all non-constant linear polynomials take negative values, $p(t)$ must have degree 2.

Since $p(t)\ge 0$ for all $t\in\mathbb{R}$, its discriminant "b^2-4ac" is not positive. Hence $4|(x,y)|^4-4|(x,y)|^2(x,x)(y,y)\le 0$, so $|(x,y)|^2\le(x,x)(y,y)$. □

4.5.2 Definition. Let A be a unital C*-algebra. We say that a linear functional $\tau\colon A\to\mathbb{C}$ is *positive* if $\tau(a)\ge 0$ for all $a\in A^+$.

4.5.3 Remark. If τ is a positive linear functional on A and $a,b\in A$ with $a\le b$ (see Remark 4.4.8) then $b-a\ge 0$ so $\tau(b-a)=\tau(b)-\tau(a)\ge 0$, so $\tau(a)\le\tau(b)$. Thus positive linear functionals are order-preserving.

4.5.4 Lemma. *Let A be a unital C*-algebra and let τ be a positive linear functional on A.*

(i). $\tau(a^*)=\overline{\tau(a)}$ *for all* $a\in A$.

(ii). $|\tau(a)|\le\tau(1)\,\|a\|$ *for all* $a\in A$. *Hence* $\tau\in A^*$ *and* $\|\tau\|=\tau(1)$.

Proof. (i) If h is a hermitian element of A then $\sigma(h)\subseteq[-t,\infty)$ for some $t\ge 0$, so $h+t1\ge 0$. Hence $\tau(h+t1)=\tau(h)+t\tau(1)\ge 0$. Since $\tau(1)\ge 0$, we must have $\tau(h)\in\mathbb{R}$. Now if $a\in A$ then $a=h+ik$ for hermitian elements h and k, so $\tau(a^*)=\tau(h-ik)=\tau(h)-i\tau(k)=\overline{\tau(h)+i\tau(k)}=\overline{\tau(a)}$.

(ii) If $a\in A$ then $a^*a\ge 0$ by Theorem 4.4.6, so $a^*a\le\|a^*a\|1=\|a\|^2 1$ by Lemma 4.4.9. Hence $\tau(a^*a)\le\tau(\|a\|^2 1)=\tau(1)\|a\|^2$.

Consider the map $(\cdot,\cdot)\colon A\times A\to\mathbb{C}$ given by $(a,b)=\tau(b^*a)$. Using (i) it is easy to check that this is a positive semi-definite sesquilinear form, so by the Cauchy-Schwarz inequality we have

$$|\tau(a)|^2=|(a,1)|^2\le(1,1)\,(a,a)=\tau(1)\,\tau(a^*a)\le\tau(1)^2\|a\|^2.$$

Hence $|\tau(a)|\le\tau(1)\|a\|$, and the result follows. □

4.5.5 Lemma. *Let A be a unital C*-algebra and let $\tau \in A^*$. Then τ is positive if and only if $\|\tau\| = \tau(1)$.*

Proof. If τ is positive then $\|\tau\| = \tau(1)$ by Lemma 4.5.4(ii). Conversely, suppose that $\|\tau\| = \tau(1)$. Multiplying τ by a positive constant if necessary, we may assume that $\|\tau\| = \tau(1) = 1$. By Lemma 4.1.8, $\tau(a)$ is real whenever a is hermitian. If $a \geq 0$ and $\|a\| \leq 1$ then $\|1 - a\| = r(1 - a) \leq 1$. Hence

$$|1 - \tau(a)| = |\tau(1 - a)| \leq 1.$$

Since $\tau(a)$ is real, this shows that $\tau(a) \geq 0$ so τ is positive. □

4.5.6 Definition. Let A be a unital C*-algebra. A *state* on A is a positive linear functional of norm 1. We write $S(A)$ for the set of states of A. Thus by the preceding lemma,

$$S(A) = \{\tau \in A^* \colon \|\tau\| = \tau(1) = 1\}.$$

4.5.7 Remark. If A is an abelian unital C*-algebra then $\Omega(A) \subseteq S(A)$ by Lemma 3.1.4. Thus states generalise the characters in the abelian case. Moreover, if we turn $S(A)$ into a topological space by giving it the subspace topology from the weak* topology on A^*, then $S(A) = J_1^{-1}(1) \cap A_1^*$ is a compact Hausdorff space.

4.5.8 Lemma. *Let A be a unital C*-algebra. If a is a hermitian element of A then there is a state $\tau \in S(A)$ with $|\tau(a)| = \|a\|$.*

Proof. Let $B = C^*(1, a)$, which is a unital abelian C*-algebra. By Corollary 3.1.11(ii), there is $\tau_0 \in \Omega(B)$ with $|\tau_0(a)| = r(a)$, and $r(a) = \|a\|$ by Proposition 4.1.12. Moreover, we have $\tau_0(1) = 1 = \|\tau_0\|$ by Lemma 3.1.4.

By the Hahn-Banach theorem [FA 3.6], we can extend τ_0 to a functional $\tau \in A^*$ with $\|\tau\| \leq 1$. Since $\tau(1) = \tau_0(1) = 1$ we have $\|\tau\| = \tau(1) = 1$ so $\tau \in S(A)$, and $\|a\| = |\tau_0(a)| = |\tau(a)|$. □

4.5.9 Lemma. *Let A be a unital C*-algebra. For every $\tau \in S(A)$ there is a Hilbert space H and a $*$-homomorphism $\pi \colon A \to \mathcal{B}(H)$ such that*

$$\tau(a^*a) \leq \|\pi(a)\|^2 \leq \|a\|^2 \quad \textit{for all } a \in A.$$

Proof. Consider the mapping $(\cdot,\cdot) \colon A \times A \to \mathbb{C}$ defined by

$$(a, b) = \tau(b^*a), \quad a, b \in A.$$

It is easy to check that this is a positive semi-definite sesquilinear form. Observe that if $a \in A$ then $a^*a \leq \|a\|^2 1$ by Lemma 4.4.9, so

$$(a, a) = \tau(a^*a) \leq \|a\|^2\tau(1) = \|a\|^2.$$

Moreover, if $a, b, c \in A$ then $(ab, c) = \tau(c^*ab) = \tau((a^*c)^*b) = (b, a^*c)$.

By the Cauchy-Schwarz inequality 4.5.1, for each $a \in A$ we have

$$(a, a) = 0 \iff (a, b) = 0 \text{ for all } b \in A.$$

Hence if $N = \{a \in A\colon (a, a) = 0\}$ then

$$N = \{a \in A\colon (a, b) = 0 \text{ for all } b \in A\}.$$

Using this expression, we can check that N is a left ideal of A. Indeed, if $a, b \in N$ then $(a + b, c) = (a, c) + (b, c) = 0$ and $(\lambda a, c) = \lambda(a, c) = 0$ and $(ca, d) = (a, c^*d) = 0$ for all $c, d \in A$ and $\lambda \in \mathbb{C}$, so $a + b$, λa and ca are in N.

Consider the vector space $V = A/N$ and let us write $[a] = a + N \in V$ for $a \in A$. We define a mapping $\langle \cdot, \cdot \rangle\colon V \times V \to \mathbb{C}$ by

$$\langle [a], [b] \rangle = (a, b), \quad a, b \in A.$$

This is well-defined since if $[a_1] = [a_2]$ and $[b_1] = [b_2]$ then $a_2 - a_1 \in N$ and $b_2 - b_1 \in N$, so $(a_1, b_1) = (a_1, b_1) + (a_2 - a_1, b_1) + (a_2, b_2 - b_1) = (a_2, b_2)$. It follows that $\langle \cdot, \cdot \rangle$ is a positive semi-definite sesquilinear form on V. In fact, it is positive definite since if $\langle [a], [a] \rangle = 0$ then $(a, a) = 0$ so $a \in N$, i.e. $[a] = 0$. Hence $(V, \langle \cdot, \cdot \rangle)$ is an inner product space.

Let H be the completion of this inner product space. It is not hard to show that $\langle \cdot, \cdot \rangle$ extends to an inner product on H turning H into a Hilbert space.

Let $\mathcal{L}(V)$ denote the set of linear maps $V \to V$, and let $\pi_0\colon A \to \mathcal{L}(V)$ be given by $\pi_0(a)[b] = [ab]$ for $[b] \in V$. This is well-defined since if $[b_1] = [b_2]$ then $b_2 - b_1 \in N$, so if $a \in A$ then $a(b_2 - b_1) = ab_2 - ab_1 \in N$ (since N is a left ideal) and so $[ab_1] = [ab_2]$. Moreover, for $a, b \in A$ we have $b^*a^*ab \leq \|a\|^2 b^*b$ by Corollary 4.4.10, so

$$\|\pi_0(a)[b]\|^2 = \|[ab]\|^2 = (ab, ab) = \tau(b^*a^*ab) \leq \|a\|^2 \tau(b^*b) = \|a\|^2 \, \|[b]\|^2.$$

Hence $\|\pi_0(a)[b]\| \leq \|a\| \, \|[b]\|$, so $\pi_0(a)$ is a continuous linear map by [FA 1.8.1], and its operator norm satisfies $\|\pi_0(a)\| \leq \|a\|$.

Thus $\pi_0(a)$ has a unique extension to a map in $\mathcal{B}(H)$, written $\pi(a)$, and $\|\pi(a)\| \leq \|a\|$. This defines a mapping $\pi\colon A \to \mathcal{B}(H)$.

We claim that π is a $*$-homomorphism. Since V is dense in H and $\pi(a)$ is continuous for all $a \in A$, it suffices to check these properties on V. For $a, b, c \in A$ and $\lambda \in \mathbb{C}$ we have

$$\pi(a + b)[c] = [(a + b)c] = [ac + bc] = \pi(a)[c] + \pi(b)[c] = \big(\pi(a) + \pi(b)\big)[c] \text{ and}$$
$$\pi(\lambda a)[c] = [\lambda ac] = \lambda\pi(a)[c] \quad \text{so } \pi \text{ is linear,}$$
$$\pi(ab)[c] = [abc] = \pi(a)[bc] = \pi(a)\pi(b)[c], \quad \text{so } \pi \text{ is a homomorphism, and}$$
$$\langle \pi(a)[b], [c] \rangle = (ab, c) = (b, a^*c) = \langle [b], \pi(a^*)[c] \rangle \text{ so } \pi(a^*) = \pi(a)^*.$$

Hence π is a $*$-homomorphism.

It remains to establish the inequality $\tau(a^*a) \leq \|\pi(a)\|^2 \leq \|a\|^2$. We have already remarked that $\|\pi(a)\| \leq \|a\|$. Now $\|[1]\|^2 = (1,1) = \tau(1) = 1$, so $\|\pi(a)\|^2 \geq \|\pi(a)[1]\|^2 = \|[a]\|^2 = (a,a) = \tau(a^*a)$. $\square$

We can now show that every C*-algebra is, up to isometric $*$-isomorphism, a subalgebra of $\mathcal{B}(H)$ for some Hilbert space H. To do so, we will piece together all of the representations constructed in Lemma 4.5.9. Since there are a lot of these representations, we will need to take care of a few technicalities first.

Let I be an index set. If we are given $\alpha_i \geq 0$ for each $i \in I$, then we will say that $(\alpha_i)_{i\in I}$ is *summable* if the set of finite sums

$$\left\{ \sum_{i\in J} \alpha_i \colon J \text{ is a finite subset of } I \right\}$$

is bounded above, and then we declare the value of $\sum_{i\in I} \alpha_i$ to be supremum of this set. If $(\alpha_i)_{i\in I}$ is summable, we write $\sum_{i\in I} \alpha_i < \infty$.

Suppose that $t_i \in \mathbb{R}$ and $\sum_{i\in I} |t_i| < \infty$. Writing $J = \{j \in I \colon t_j \geq 0\}$ we can define $\sum_{i\in I} t_i = \sum_{j\in J} t_j - \sum_{k\in I\setminus J}(-t_k)$. Similarly, if $\lambda_i \in \mathbb{C}$ and $\sum_{i\in I} |\lambda_i| < \infty$ then we can define $\sum_{i\in I} \lambda_i = \sum_{i\in I} \mathrm{Re}(\lambda_i) + i \sum_{i\in I} \mathrm{Im}(\lambda_i)$.

If $\{H_i \colon i \in I\}$ is a family of Hilbert spaces, then we can define an inner product space H as follows:

$$H = \left\{ (\xi_i)_{i\in I} \colon \xi_i \in H_i \text{ for each } i \in I, \text{ and } \sum_{i\in I} \|\xi_i\|^2 < \infty \right\}$$

with "pointwise" vector space operations $(\xi_i)_{i\in I} + (\eta_i)_{i\in I} = (\xi_i + \eta_i)_{i\in I}$, $\lambda(\xi_i)_{i\in I} = (\lambda\xi_i)_{i\in I}$ and inner product

$$\langle (\xi_i)_{i\in I}, (\eta_i)_{i\in I} \rangle = \sum_{i\in I} \langle \xi_i, \eta_i \rangle.$$

It follows from the Cauchy-Schwarz inequality that $\sum_{i\in I} |\langle \xi_i, \eta_i \rangle| < \infty$ if $(\xi_i)_{i\in I}$, $(\eta_i)_{i\in I} \in H$, so the inner product is well-defined.

It is not too hard to show that H is then a complete inner product space, i.e. H is a Hilbert space. We usually write

$$H = \bigoplus_{i\in I} H_i$$

and call H the *Hilbertian direct sum* of the Hilbert spaces $\{H_i \colon i \in I\}$.

Suppose that $a_i \in \mathcal{B}(H_i)$ for each $i \in I$ and that $\sup_{i\in I} \|a_i\| < \infty$. We may define an operator $a \in \mathcal{B}(H)$ by $a((\xi_i)_{i\in I}) = (a(\xi_i))_{i\in I}$ for $(\xi_i)_{i\in I} \in H$. Note that $\sum_{i\in I} \|a_i\xi_i\|^2 \leq (\sup_{i\in I} \|a_i\|) \sum_{i\in I} \|\xi_i\|^2$ for each $x \in X$, so a is a bounded operator, with $\|a\| \leq \sup_{i\in I} \|a_i\|$. In fact, it is easy to show that $\|a\| = \sup_{i\in I} \|a_i\|$ [exercise]. We will write $a = \bigoplus_{i\in I} a_i$.

4.5.10 Theorem (The Gelfand-Naimark-Segal theorem)**.** *If A is a unital C^*-algebra then A is isometrically $*$-isomorphic to a subalgebra of $\mathcal{B}(H)$ for some Hilbert space H.*

Proof. Given $\tau \in S(A)$, let us write H_τ and $\pi_\tau \colon A \to B(H_\tau)$ for the Hilbert space and the $*$-homomorphism obtained from Lemma 4.5.9. Let

$$H = \bigoplus_{\tau\in S(A)} H_\tau \quad \text{and define} \quad \pi(a) = \bigoplus_{\tau\in S(A)} \pi_\tau(a) \quad \text{for } a \in A.$$

It is easy to see that this defines a $*$-homomorphism $\pi \colon A \to \mathcal{B}(H)$. If $a \in A$ then since $\|\pi(a)\| = \sup_{\tau\in S(A)} \|\pi_\tau(a)\|$, we have

$$\sup_{\tau\in S(A)} \tau(a^*a) \leq \|\pi(a)\|^2 \leq \|a\|^2$$

by Lemma 4.5.9. However, a^*a is hermitian so by Lemma 4.5.8,

$$\sup_{\tau\in S(A)} \tau(a^*a) \geq \|a^*a\| = \|a\|^2.$$

Putting these inequalities together shows that $\|\pi(a)\| = \|a\|$ for each $a \in A$, so π is an isometric $*$-homomorphism. In particular, the range of π is a closed $*$-subalgebra of $\mathcal{B}(H)$ which is $*$-isomorphic to A. $\square$