442 Tutorial Sheet 5 Solutions[1]

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1. (2) This question is revision of Lagrangian mechanics. By varying the field derive the Euler-Lagrance equations from $\delta S = 0$ where

$$S = \int L(t, \mathbf{q}, \dot{\mathbf{q}}) dt \tag{1}$$

Solution: Well vary $\mathbf{q}$ to get

$$\delta L(t, \mathbf{q}, \dot{\mathbf{q}}) = \sum_i \frac{\partial L}{\partial q_i} \delta q_i + \sum_i \frac{\partial L}{\partial \dot{q}_i} \delta \dot{q}_i \tag{2}$$

or

$$\delta S = \int \left(\sum_i \frac{\partial L}{\partial q_i} \delta q_i + \sum_i \frac{\partial L}{\partial \dot{q}_i} \delta \dot{q}_i \right) dt \tag{3}$$

and then use integration by part on the second term, assuming that there are fixed boundary conditions there is no boundary term and

$$\delta S = \int \left(\sum_i \frac{\partial L}{\partial q_i} - \sum_i \frac{d}{dt} \frac{\partial L}{\partial \dot{q}_i} \right) \delta q_i dt \tag{4}$$

and if this is true for all variations satisfying suitable boundary conditions, we get the Euler-Lagrange equation

$$\sum_i \frac{\partial L}{\partial q_i} - \sum_i \frac{d}{dt} \frac{\partial L}{\partial \dot{q}_i} = 0 \tag{5}$$

2. (2) Derive the Euler-Largrange equation

$$\partial_\mu \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)} - \frac{\partial \mathcal{L}}{\partial \phi} = 0 \tag{6}$$

from $\delta S = 0$ where

$$S = \int \mathcal{L}(t, \phi, \partial_\mu \phi) d^4 x \tag{7}$$

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Solution: Well this is almost the same, vary ϕ and we get, by Taylor expanding $\mathcal{L}$

$$\delta S = \int \left(\frac{\partial \mathcal{L}}{\partial \phi} \delta \phi + \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)} \partial_\mu \delta \phi \right) d^4 x \tag{8}$$

and then use a Gauss law and zero boundary conditions to give

$$\delta S = \int \left(\frac{\partial \mathcal{L}}{\partial \phi} - \frac{\partial}{\partial x^\mu} \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)} \right) \delta \phi d^4 x \tag{9}$$

and arbitrarity of $\delta\phi$ gives the Euler-Lagrange equations.

3. (3) Derive the Euler-Largange equation

$$\nabla_\mu \frac{\partial \mathcal{L}}{\partial(\nabla_\mu \phi)} - \frac{\partial \mathcal{L}}{\partial \phi} = 0 \tag{10}$$

from $\delta S = 0$ where

$$S = \int \mathcal{L}(t, \phi, \nabla_\mu \phi) \sqrt{g} d^4 x \tag{11}$$

Recall that $\nabla_\mu \phi = \partial_\mu \phi$ and that $\sqrt{g} \nabla_\mu X^\mu = \partial_\mu \sqrt{g} X^\mu$.

Solution: This one isn't really so different, bearing in mind $\nabla_\mu \phi = \partial_\mu \phi$ nothing changes until we use the Gauss law, then we get

$$\delta S = \int \left(\frac{\partial \mathcal{L}}{\partial \phi} \sqrt{g} - \frac{\partial}{\partial x^\mu} \sqrt{g} \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)} \right) \delta \phi d^4 x \tag{12}$$

and then commute the $\sqrt{g}$ through to give

$$\delta S = \int \left(\frac{\partial \mathcal{L}}{\partial \phi} \sqrt{g} - \sqrt{g} \nabla_\mu \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)} \right) \delta \phi d^4 x \tag{13}$$

and rewrite the $\partial_\mu \phi$s as $\nabla_\mu \phi$ to get the answer.

4. (4) Calculate the energy-momentum tensor

$$T_{\mu\nu} = F_{\lambda\mu} F^\lambda{}_\nu - \frac{1}{4} g_{\mu\nu} F_{\lambda\rho} F^{\lambda\rho} \tag{14}$$

for the Maxwell field:

$$S = -\frac{1}{4} \int F^{\mu\nu} F_{\mu\nu} \sqrt{g} d^4 x \tag{15}$$

where

$$F_{\mu\nu} = \nabla_\mu A_\nu - \nabla_\nu A_\mu \tag{16}$$

Solution: So, we need to write the action in a way that exposes the dependence on the metric,

$$S = -\frac{1}{4} \int g^{\lambda\mu} g^{d\nu} F_{\lambda\rho} F_{\mu\nu} \sqrt{g} d^4x \tag{17}$$

and then use

$$\begin{aligned} \delta g_{\mu\nu} &= h_{\mu\nu} \\ \delta g^{\mu\nu} &= -h^{\mu\nu} \\ \delta\sqrt{g} &= \frac{1}{2}\sqrt{g} h \end{aligned} \tag{18}$$

where, as usual, we are using the notation $h = g^{\mu\nu} h_{\mu\nu}$. Now,

$$S = \int \mathcal{L} d^4x \tag{19}$$

so that

$$\begin{aligned} \delta S &= -\int \left(-\frac{1}{2} h^{\lambda\mu} g^{\rho\nu} F_{\lambda\rho} F_{\mu\nu} + \frac{1}{8} g^{\lambda\mu} g^{\rho\nu} F_{\lambda\rho} F_{\mu\nu} h \right) \sqrt{g} d^4x & (20) \\ &= \int \left(\frac{1}{2} F_{\lambda\rho} F_{\mu}{}^{\rho} - \frac{1}{8} F^{\rho\nu} F_{\rho\nu} g_{\lambda\mu} \right) h^{\lambda\mu} \sqrt{g} d^4x & (21) \end{aligned}$$

hence

$$\begin{aligned} T_{\lambda\mu} &= 2 \frac{\delta S}{\delta h^{\lambda\mu}} \\ &= F_{\lambda\rho} F_{\mu}{}^{\rho} - \frac{1}{4} F^{\rho\nu} F_{\rho\nu} g_{\lambda\mu} \end{aligned} \tag{22}$$