442 Sample Paper[1]

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This is a sample paper intended to give an indication as to the style of the paper, howerver, I haven't made up new questions so they are all off the Problem Sheets, in the exam this won't be the case, in the exam roughly half the non-bookwork material will be original and half will have been in a problem sheet, perhaps with small differences.

1. (a) Define geodesic coordinates and writing the Riemann tensor in terms of these coordinates show $R_{abcd} = R_{cdab}$. [BOOK WORK]

   (b) Find the scalar curvature on a two-dimensional hyperboloid:

   $$x^2 + y^2 - t^2 = -r^2 \tag{1}$$

   embedded in the Minkowski space

   $$ds^2 = -dt^2 + dx^2 + dy^2 \tag{2}$$

2. Define a geodesic and prove that the geodesic gives the shortest path between two points. [BOOK WORK] Find the time-like geodesics for the metric

   $$ds^2 = \frac{1}{t^2}\left(-dt^2 + dx^2\right) \tag{3}$$

   You might want to use the integral

   $$\int \frac{dt}{t\sqrt{1+C^2t^2}} = \frac{1}{2}\log\left(\frac{\sqrt{1+C^2t^2}-1}{\sqrt{1+C^2t^2}+1}\right) \tag{4}$$

3. In Einstein's theory of gravity a test particle moves on a geodesic in a space-time whose curvature is determined by the Einstein equation. In Newton's theory of gravity a test particle has an acceleration according to Newton's law with the force determined by the Poisson equation. Show that in the non-relativistic weak-field approximation the Einsteinian gravity for a static space-time is approximated by Newtonian gravity. [BOOK WORK]

4. Let

   $$g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu} \tag{5}$$

   where, by a choice of coordinates, $h_{\mu\nu}$ satisfies the harmonic gauge condition:

   $$\partial_\mu h^\mu_\nu = \frac{1}{2}\partial_\nu h \tag{6}$$

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show that the Einstein equations reduce to

$$-\frac{1}{2}\partial^2 h_{\mu\nu} + \frac{1}{4}\eta_{\mu\nu}\partial^2 h = 8\pi T_{\mu\nu} \tag{7}$$

where, as usual, $h = h^\mu_\mu$ is the trace. By taking the trace of both sides and solving for $\partial^2 h$, show that this can be written as

$$\partial^2 h_{\mu\nu} = -16\pi\left(T_{\mu\nu} - \frac{1}{2}\eta_{\mu\nu}T\right) \tag{8}$$

where $T = \eta^{\mu\nu}T_{\mu\nu}$.

5. Calculate the energy-momentum tensor

$$T_{\mu\nu} = F_{\rho\mu}F^{\rho}{}_{\nu} - \frac{1}{4}g_{\mu\nu}F_{\lambda\rho}F^{\lambda\rho}$$

for the Maxwell field:

$$S = -\frac{1}{4}\int F^{\mu\nu}F_{\mu\nu}\sqrt{g}d^4x$$

where

$$F_{\mu\nu} = \nabla_\mu A_\nu - \nabla_\nu A_\mu$$

where you may quote the variation $\delta g = gh/2$ for $\delta g_{\mu\nu} = h_{\mu\nu}$ and $h = \eta^{\mu\nu}h_{\mu\nu}$. Verify that this expression satisfies the identity $\nabla^\mu T_{\mu\nu} = 0$.

6. Show that a universe with $\Omega_0 = 1 - \epsilon$ for small positive ϵ has age

$$t_0 = \frac{2}{3H_0}\left(1 + \frac{1}{5}\epsilon\right) + O(\epsilon^2).$$

You might find the expansion

$$\sinh^{-1} x = x - \frac{1}{6}x^3 + \frac{3}{40}x^5$$

useful, you might also benefit from being reminded that the usual way of integrating

$$\int \frac{dx}{\sqrt{\frac{A}{x} + 1}}$$

is to substitute $x = A\sinh^2\theta$ and then use

$$\int d\theta \sinh^2\theta = \frac{1}{2}\sinh\theta\cosh\theta - \frac{1}{2}\theta$$

[BOOK WORK]

7. The equation of state is often written in *adiabatic form*

$$p = (\gamma - 1)\rho$$

where p is pressure and ρ is density and $0 \leq \gamma \leq 2$ is the *adiabatic index* with $\gamma = 1$ for dust and $\gamma = 4/3$ for radiation.

(a) Calculate $\rho(a)$ for general γ. Find the age of the universe for $k = 0$.

(b) Find γ so that the expansion rate is constant. With this value of γ find $a(t)$ for $k = 1$ and $k = -1$.

(c) In the same notation, show

$$\dot{\Omega} = (2 - 3\gamma)H\Omega(1 - \Omega)$$

Define the *logarithmic scale factor* $s = \log a$ and write an equation for $d\Omega/ds$.

(d) Comment on the flatness problem.

8. Consider a simplified model of the history of a flat universe involving a period of inflation. The history is split into four periods: (a) $0 < t < t_3$ radiation only; (b) $t_3 < t < t_2$ vacuum energy dominates with an effective cosmological constant $\Lambda = 3t_3^2/4$; (c) $t_2 < t < t_1$ a period of radiation domination; (d) $t_1 < t < t_0$ matter domination.

(a) Show that in (c) $\rho(t) = \rho_r(t) = 3\pi t^2/32$ and in (d) $\rho(t) = \rho_m(t) = \pi t^2/6$. The functions ρ_r and ρ_m are introduced for later convenience.

(b) Give simple analytic formulas for $a(t)$ which are approximately true in these four epochs.

(c) Shat that during the inflationary epoch the universe expands by a factor

$$\frac{a(t_2)}{a(t_3)} = \exp\left(\frac{t_2 - t_3}{2t_3}\right) \tag{9}$$

(d) In the notation introduced earlier, show

$$\frac{\rho_r(t_0)}{\rho_m(t_0)} = \frac{9}{16}\left(\frac{t_1}{t_0}\right)^{2/3} \tag{10}$$

(e) If $t_3 = 10^{-35}$ seconds, $t_2 = 10^{-32}$ seconds, $t_1 = 10^4$ years and $t_0 = 10^{10}$ years, give a sketch of $\log a$ against $\log t$ marking any important epochs.

9. Kaluza-Klein question, will be added later.