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HOLOMORPHIC EXTENSION OF SMOOTH CR-MAPPINGS BETWEEN REAL-ANALYTIC AND REAL-ALGEBRAIC CR-MANIFOLDS *

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1. Introduction and results. The classical Schwarz reflection principle states that a continuous map f between real-analytic curves M and M' in $\mathbb{C}$ that locally extends holomorphically to one side of M, extends also holomorphically to a neighborhood of M in $\mathbb{C}$. It is well-known that the higher-dimensional analog of this statement for maps $f\colon M \to M'$ between real-analytic CR-submanifolds $M \subset \mathbb{C}^N$ and $M' \subset \mathbb{C}^{N'}$ does not hold without additional assumptions (unless M and M' are totally real). In this paper, we assume that f is $\mathcal{C}^\infty$-smooth and that the target M' is *real-algebraic*, i.e. contained in a real-algebraic subset of the same dimension. If f is known to be locally holomorphically extendible to one side of M (when M is a hypersurface) or to a wedge with edge M (when M is a generic submanifold of higher codimension), then f automatically satisfies the tangential Cauchy-Riemann equations, i.e. it is CR. On the other hand, if M is *minimal*, any CR-map $f\colon M \to M'$ locally extends holomorphically to a wedge with edge M by TUMANOV's theorem [Tu88] and hence, in that case, the extension assumption can be replaced by assuming f to be CR.

Local holomorphic extension of a CR-map $f\colon M \to M'$ may clearly fail when M' contains an (irreducible) complex-analytic subvariety E' of positive dimension and $f(M) \subset E'$. Indeed, any nonextendible CR-function on M composed with a nontrivial holomorphic map from a disc in $\mathbb{C}$ into E' yields a counterexample. Our first result shows that this is essentially the only exception. Denote by $\mathcal{E}'$ the set of all points $p' \in M'$ through which there exist irreducible complex-analytic subvarieties of M' of positive dimension. We prove:

THEOREM 1.1. *Let $M \subset \mathbb{C}^N$ and $M' \subset \mathbb{C}^{N'}$ be respectively connected real-analytic and real-algebraic CR-submanifolds. Assume that M is minimal at a point $p \in M$. Then for any $\mathcal{C}^\infty$-smooth CR-map $f\colon M \to M'$, at least one of the following conditions holds:*

(i) f extends holomorphically to a neighborhood of p in $\mathbb{C}^N$;

(ii) f sends a neighborhood of p in M into $\mathcal{E}'$.

If M' is a real-analytic hypersurface, the set $\mathcal{E}'$ consists exactly of those points that are not of finite type in the sense of D'ANGELO [D'A82] (see LEMPERT [L86] for the proof) and, in particular, $\mathcal{E}'$ is closed. The same fact also holds if M' is any real-analytic submanifold or even any real-analytic subvariety (see [D'A91]). However, in general, $\mathcal{E}'$ may not even be a real-analytic subset (see Example 2.1). In case $\mathcal{E}' = V'$ is a subvariety, we have:

COROLLARY 1.2. *Let $M \subset \mathbb{C}^N$ and $M' \subset \mathbb{C}^{N'}$ be as in Theorem* 1.1. *Assume that M is minimal at a point $p \in M$ and that all positive-dimensional irreducible complex-analytic subvarieties in M' are contained in a fixed* (*complex-analytic*) *subvariety $V' \subset$*

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