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Topology 44 (2005) 557–584

TOPOLOGY

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Equivalence and embedding problems for CR-structures of any codimension

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Received 22 April 2003; received in revised form 7 June 2004; accepted 18 November 2004

Abstract

An approach is suggested to equivalence and embedding problems for smooth CR-submanifolds of complex spaces (and, more generally, for abstract CR-manifolds) in terms of complete differential systems in jet bundles satisfied by all CR-equivalences or CR-embeddings respectively. For equivalence problems, manifolds are assumed to be of finite type and finitely nondegenerate. These are higher order generalizations of the corresponding nondegeneracy conditions for the Levi form. It is shown by a simple example that these nondegeneracy conditions cannot be even slightly relaxed to more general known conditions. In particular, for essentially finite hypersurfaces in $\mathbb{C}^2$, such a complete system does not exist in general. For embedding problems, source manifolds are assumed to be of finite type and their embeddings to be finitely nondegenerate. Sufficient conditions on CR-manifolds are given, where the last condition is automatically satisfied by all CR-embeddings.

Keywords: CR-structure; Equivalence problem; Nondegeneracy conditions; Complete systems; Jet parametrization

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[1] The first author was supported by postdoctoral fellowship program from Korea Science and Engineering Foundation (KOSEF).
[2] The second author is supported in part by a grant from the Italian Consiglio Nazionale delle Ricerche.

0040-9383/$ - see front matter
doi:10.1016/j.top.2004.11.004

1. Introduction

1.1. The scope of the paper

One of the main results of this paper provides an approach to the solution of the equivalence problem for ($\mathscr{C}^\kappa$) CR-structures having nondegenerate Levi forms or, more generally, satisfying nondegeneracy conditions in terms of "higher order Levi forms". The *equivalence problem*, that goes back to Poincaré [52] and Cartan [19], can be stated as follows:

Given two manifolds with CR-structures, when does there exist a CR-diffeomorphism between them?

This problem is an important case of the more general equivalence problem for the so-called *geometric structures* (see e.g. [54]) introduced by Cartan. It is also closely related to the biholomorphic equivalence problem for bounded domains in $\mathbb{C}^N$, $n \geqslant 2$. Indeed, by Fefferman's theorem [32] (and Bochner–Hartogs theorem for the converse), two strongly pseudoconvex smoothly bounded domains are biholomorphically equivalent if and only if their boundaries have equivalent CR-structures.

Among others, two major approaches are known for the equivalence problem: the one of *normal forms* and the one of *differential systems*. The normal form approach applies to CR-submanifolds of $\mathbb{C}^N$ (i.e. for embedded CR-structures) and concerns with normalization of the Taylor series of their defining equations at a given point (see e.g. the survey [6] for the references). Once a normal form is established, it solves the local biholomorphic equivalence problem for real-analytic CR-structures. Another approach to the equivalence problem for real-analytic hypersurfaces in $\mathbb{C}^n$ satisfying higher order nondegeneracy assumptions is due to Baouendi et al. [2].

In this paper we follow the approach of differential systems. Pioneer work here was done by Cartan [19] who gave a solution of the equivalence problem for smooth *Levi-nondegenerate* hypersurfaces in $\mathbb{C}^2$ and later by Tanaka [55–57], Chern and Moser [23], Burns and Shnider [17,18], Jacobowitz [41] and Webster [58] in $\mathbb{C}^N$, where the problem has been reduced to the equivalence problem for the so-called *absolute parallelisms*. The latter means a choice of a distinguished frame in each tangent space. The reason that the equivalence problem for absolute parallelisms is well-understood (see e.g. [54]) is that the condition for a diffeomorphism f to preserve an absolute parallelism can be reduced to ordinary differential equations for f along smooth real curves. Further reductions to absolute parallelisms have been obtained by Mizner [51], Garrity and Mizner [34], Ezhov et al. [30], Čap and Schichl [20], Schmalz and Slovak [53], Čap et al. [22] and Čap and Schmalz [21] for certain Levi-nondegenerate CR-manifolds of higher codimension and by Ebenfelt [28] for certain uniformly Levi-degenerate hypersurfaces in $\mathbb{C}^3$. In each case, a stronger condition than that of Levi-nondegeneracy (or that of finite nondegeneracy in the latter case) had to be imposed. The key problem for general Levi-nondegenerate CR-structures of higher codimension, when trying to use the known methods, is the fact that already the first prolongation of the corresponding differential system may develop singularities that are not allowed in general theory (see e.g. [16]). When extending this theory to CR-manifolds of higher codimension, one has the additional difficulty that the automorphism group of the Levi form may change even in its dimension as the reference point varies.

In the present paper we follow a "rougher" approach which, however, will allow us to treat more general cases of CR-structures, in particular, all Levi-nondegenerate CR-structures in any codimension. Instead of reducing CR-equivalence problem to that for absolute parallelisms, we follow a more direct way of reducing it to solving (ordinary differential equations) ODEs. We further treat the case, where the Levi form may change its rank from point to point. To the authors' knowledge, no reduction to

absolute parallelism is known in this case. A natural example of such a CR-structure is given by a connected oriented hypersurface in $\mathbb{C}^N$ that has both strongly pseudoconvex and strongly pseudoconcave points. Then it is clear from the continuity of the Levi form that there must exist also Levi-degenerate points. It is also clear that such points cannot be avoided even by (global) small perturbation of the hypersurface.

The reduction to ODEs is obtained here by deriving the so-called *complete systems*. A partial differential equations PDE system is called *complete* if it expresses all partial derivatives of the highest order as functions of the lower order derivatives. In Section 1.3 we state Theorem 1.1 establishing a complete system for CR-diffeomorphisms and explain in more details how it can be used for the equivalence problem. Example 1.5 shows that the nondegeneracy conditions in Theorem 1.1 cannot be replaced even by the next weaker known general conditions. In Section 1.4 an analogous statement yields an approach to an *embedding problem* asking when there exists a CR-embedding with certain properties of a given CR-manifold into another one, which is a natural generalization of the equivalence problem. The reader is referred to that paragraph for more detailed discussion of this problem. Then in Section 2 we state results on the existence of complete systems in a more general context containing both diffeomorphisms and embeddings as well as some other situations.

The general approach of complete system has been successfully used for other problems related to CR-manifolds [29,37,38,43] and to related problems in differential geometry [24,25,36,42]. In fact, the present paper was very much inspired by the work of Ebenfelt [29] (that was, in turn, inspired by the work of Han [37]), where complete systems were used to obtain local jet parametrizations and unique determinations of CR-diffeomorphisms for smooth finitely nondegenerate hypersurfaces in $\mathbb{C}^N$ (see below for the definition). On the other hand, both methods of [37] and of [29] to derive complete systems are very different from ours and yield in their cases of applicability systems of higher order than that given by Theorems 1.1 and 2.1 below. One of the consequences of this differences is the fact that the order in our system is independent of the (Kohn–Bloom–Graham) type of the CR-manifold (see below) and hence, such a system cannot be obtained by "gaining missing directions through commutators" as in the previous work.

The method to derive complete systems in present paper is based on a development of the *Segre set method* (originally due to Baouendi et al. [1]) and is further inspired by the techniques used in [2,61] and especially [4,11]. In [4] the authors obtain results on Taylor series of CR-maps between smooth CR-manifolds at a given point. However, the present paper is the first one, where extensions of techniques mentioned above are used to obtain properties of CR-maps of smooth CR-manifolds "beyond" their Taylor series. Other applications of the Segre sets method include a characterization of the finite type [1], unique determination and an (explicit algorithmic) holomorphic parametrization of CR-maps by their jets [2,4,43,48,61], algebraicity of (real-analytic) CR-maps between real-algebraic CR-manifolds [1,62] and convergence properties of formal maps [5,8,11,48–50] (see also the references in the quoted papers about related results obtained by other methods).

The construction of the Segre sets cannot be directly extended neither to abstract non-embeddable (into some $\mathbb{C}^N$) nor to smoothly embeddable CR-structures because it makes use of the complexified defining function: if a submanifold of $\mathbb{C}^N$ is locally defined by vanishing of a power series $\rho(z,\overline{z})$, the complexification $\rho(z,\overline{w})$ with $z \neq w$ is needed already in the definition of the *Segre varieties* (that are the "first order Segre sets") $Q_w := \{z : \rho(z,\overline{w}) = 0\}$. The Segre sets are then defined inductively by $Q_w^1 := Q_w$, $Q_w^{s+1} := \cup_{z\in Q_w^s} Q_z$. Because of this analytic nature of the Segre sets, the above mentioned applications of the Segre set method cannot be directly extended to the case of smooth CR-structures

considered in this paper. In this paper we develop *approximate Segre sets method* that is applied to any smooth CR-submanifold of $\mathbb{C}^N$ and, more generally, to any abstract smooth CR-structure.

We would like to mention that the approximate Segre sets method in this paper yields a complete system in an explicit and algorithmic way. In particular, CR-manifolds of finite smoothness are treated with explicit estimates for the required initial regularity in terms of their "degree of nondegeneracy".

1.2. Levi-nondegeneracy and higher order nondegeneracies

We begin with CR-submanifolds of $\mathbb{C}^N$ that serve as basic examples of (abstract) CR-manifolds. A submanifold $M \subset \mathbb{C}^N$ is called CR if its complex tangent space $T_p^{\mathrm{c}}M := T_pM \cap JT_pM$ has constant dimension for $p \in M$, where $J: T\mathbb{C}^N \to T\mathbb{C}^N$ denotes the standard complex structure. A (1, 0) (resp. (0, 1)) vector on M is a complex tangent vector of the form $X - \mathrm{i}JX$ (resp. $X + \mathrm{i}JX$) for $X \in T^{\mathrm{c}}M$ and a (1, 0) (resp. (0, 1)) vector field is a section of the subbundle $T^{1,0}M \subset T^{\mathrm{c}}M \otimes \mathbb{C}$ (resp. $T^{0,1}M \subset T^{\mathrm{c}}M \otimes \mathbb{C}$) of all (1, 0) (resp. (0, 1)) vectors. If M is of class $\mathscr{C}^\kappa$, its (1, 0) and (0, 1) vector fields can be taken of class $\mathscr{C}^{\kappa-1}$. The best known invariant for CR-manifolds is the *Levi form* defined to be the (unique) hermitian form

$$\mathscr{L}_p: T_p^{0,1}M \times T_p^{0,1}M \to (T_pM/T_p^{\mathrm{c}}M) \otimes \mathbb{C} \tag{1.1}$$

satisfying

$$\mathscr{L}_p(L_1(p), L_2(p)) = \frac{1}{2\mathrm{i}}\pi[L_1, \overline{L}_2](p), \tag{1.2}$$

for all (0, 1) vector fields L_1, L_2, where $\pi: TM \otimes \mathbb{C} \to (TM/T^{\mathrm{c}}M) \otimes \mathbb{C}$ is the canonical projection. According to Beloshapka [13], the Levi form $\mathscr{L}_p$ is called *nondegenerate* if (1) $\mathscr{L}_p(L_1, L_2) = 0$ for all L_2 implies $L_1 = 0$ and (2) the vectors $\mathscr{L}_p(L_1, L_2)$ for all L_1 and L_2 span the vector space $(T_pM/T_p^{\mathrm{c}}M) \otimes \mathbb{C}$. In [13] it was shown that, if M is a real quadric of the form $\{\mathrm{Im}\, w = H(z, z)\}$, where $H: \mathbb{C}^m \times \mathbb{C}^m \to \mathbb{C}^d$ is hermitian, this condition of nondegeneracy is necessary and sufficient for the local stability group (i.e. for the group of local biholomorphisms of $\mathbb{C}^{m+d}$ fixing a point) of M to be a (finite-dimensional) Lie group.

For manifolds more general than quadrics, the nondegeneracy of the Levi form appears to be only a sufficient condition for the finite-dimensionality of the local stability group. For instance, the tube over the light cone in $\mathbb{C}^n$ given by $(\mathrm{Re}\, z_n)^2 = (\mathrm{Re}\, z_1)^2 + \cdots + (\mathrm{Re}\, z_{n-1})^2$ is everywhere Levi-degenerate but has a finite-dimensional stability group. In order to have more optimal conditions, one often has to replace Levi-nondegeneracy by finer *higher order* nondegeneracy conditions (see e.g. surveys [6,63] for related references). Natural generalizations of the conditions (1) and (2) in the above definition of nondegenerate Levi form are the conditions of finite type and of finite nondegeneracy.

Recall that a CR-submanifold M is said to be of *finite type* at a point $p \in M$ (in the sense of Kohn [45] and Bloom and Graham [14]) if all (1, 0) and (0, 1) vector fields on M together with all their higher order commutators span the space $T_pM \otimes \mathbb{C}$. The minimal length $\nu \geqslant 1$ of the commutators required to span $T_pM \otimes \mathbb{C}$ is called the *type* of M at p and is denoted here by ν. (This is not a well-established terminology but it will be used throughout this paper.) The *type of M* (without specifying a point) is the maximal type of M at p for all $p \in M$. If the latter number is finite, M is said to be simply of *finite type*.

A CR-submanifold $M \subset \mathbb{C}^N$ is called *finitely nondegenerate* at p (see [7,27,35] and also [3, Section 11.1]) if, for some integer $k \geqslant 1$,

$$\operatorname{span}_{\mathbb{C}}\{\mathscr{T}_{L_s}(\ldots\mathscr{T}_{L_2}(\mathscr{T}_{L_1}\theta)\ldots)(p) : 0 \leqslant s \leqslant k,\ \ L_j \in \Gamma(T^{0,1}M), \theta \in \Gamma(T^{*0}M)\} = T_p^{*1,0}M, \tag{1.3}$$

where $T^{*0}M \subset T^{*1,0}M$ denote the bundles of complex 1-forms on M that vanish respectively on $T^{\mathrm{c}}M \otimes \mathbb{C}$ and on $T^{0,1}M \subset T^{\mathrm{c}}M \otimes \mathbb{C}$ and $\mathscr{T}_L = \mathrm{i}_L \circ d + d \circ \mathrm{i}_L$ is the Lie derivative along L. Recall that for any $(0,1)$ vector field L, the Lie derivative $\mathscr{T}_L$ leaves the space $\Gamma(T^{*1,0}M)$ invariant and is given there by $\mathscr{T}_L\omega = \mathrm{i}_L d\omega$, where i_L denotes the contraction. If l is the minimum of all k for which (1.3) holds, M is called *l-nondegenerate* at p. In this case we also say that l is the *degeneracy* of M at p. (Again, also this is not an established terminology.) The *degeneracy of M* is the maximal degeneracy of M at p for all $p \in M$. If the latter number is finite, M is said to be *finitely nondegenerate*. The reader is referred to the book [3] for further details.

Both conditions to have type ν and degeneracy l at p involve finite number of differentiations and therefore are meaningful for $\mathscr{C}^\kappa$ CR-submanifolds provided $\nu \leqslant \kappa$ and $l \leqslant \kappa - 1$. In contrast to the Levi-nondegeneracy, the condition of finite nondegeneracy and of finite type (at every point) can be always achieved by (global) small perturbations of the submanifold (see the forthcoming paper [12]).

1.3. Complete systems for CR-diffeomorphisms

In this section, for the sake of simplicity of statements, we restrict ourselves to the case of smooth ($\mathscr{C}^\infty$) CR-manifolds. More general statements for finite smoothness will be given in Section 2. Recall that an *abstract CR-manifold of CR-dimension n and of CR-codimension d* is a real $(2n+d)$-dimensional (smooth) manifold M together with a real vector subbundle $T^{\mathrm{c}}M \subset TM$ of real rank $2n$ and a complex structure J viewed as a bundle automorphism $J\colon T^{\mathrm{c}}M \to T^{\mathrm{c}}M$ with $J^2 = -\mathrm{id}$ satisfying the following *integrability condition*: for any two $(0,1)$ vector fields, their commutator is also a $(0,1)$ vector field (equivalently, a commutator of $(1,0)$ vector fields is $(1,0)$). Here $(1,0)$ and $(0,1)$ vector fields, finite type and finite nondegeneracy are defined by repeating the definitions in the embedded case given above. See e.g. [3,15] for further details on abstract CR manifolds.

Given two manifolds M and M' and an integer k, denote by $J^k(M, M')$ the bundle of all k-jets of smooth maps f between (open pieces of) M and M' (at all points) and by $j_x^k f \in J^k(M, M')$ the k-jet of f at $x \in M$. If, in addition, $\dim M = \dim M'$, denote by $G^k(M, M') \subset J^k(M, M')$ the open subset of all jets of invertible maps. One of the main results of this paper providing, in particular, an approach to the equivalence problem, yields complete systems in jet bundles satisfied by all CR-diffeomorphisms:

Theorem 1.1. *Let M and M' be smooth (abstract) CR-manifolds of the same dimension, both finitely nondegenerate and of finite type. Then there exist a number r and a smooth map $\Phi\colon G^r(M, M') \to G^{r+1}(M, M')$ such that every smooth CR-diffeomorphism f between open pieces of M and M' satisfies the complete differential system*

$$j_x^{r+1} f = \Phi(j_x^r f), \tag{1.4}$$

for all x in the domain of definition of f. The number r can be chosen to be $2(d+1)l$, where d is the CR-codimension and l is the degeneracy of M. The map Φ can be chosen as a section, i.e. satisfying $\pi \circ \Phi = \mathrm{id}$, where $\pi\colon G^{r+1}(M, M') \to G^r(M, M')$ is the natural projection.

Remark. The assumptions of Theorem 1.1 are, in particular, satisfied for Levi-nondegenerate CR-manifolds (see e.g. [3]). Since both conditions of finite type and finite nondegeneracy are invariant under CR-diffeomorphisms, it is sufficient to require them only for M or only for M'. Observe that the order r of the system (1.4) does not depend on the type of M and M' (defined above).

Theorem 1.1 and Corollaries 1.2–1.3 in this paragraph are special cases of the more general statements given by Theorem 2.1 and Corollaries 2.2–2.3 below.

In case M and M' are hypersurfaces in $\mathbb{C}^N$, a conclusion similar to that of Theorem 1.1 has been recently obtained by Ebenfelt [29] with the order $r = l^3 + l^2 + l + 1$ (instead of $r = 4l$ given by Theorem 1.1 in the hypersurface case ($d = 1$)).

As an application of Theorem 1.1 we obtain the following unique determination result:

Corollary 1.2. *Let M, M' and r be as in Theorem* 1.1. *Then CR-diffeomorphisms between connected open pieces of M and M' are uniquely determined by their r-jets at any point of M, i.e. if $f, g: U \subset M \to M'$ are CR-diffeomorphisms, U is connected, $p \in U$ and $j^r_p f = j^r_p g$, then $f \equiv g$.*

Furthermore, we obtain a local jet-parametrization, meaning that the CR-diffeomorphisms are not only uniquely determined by their r-jets but also depend on them in a smooth fashion. We denote by $G^r_{p,p'}(M, M')$ the fiber in $G^r(M, M')$ over $(p, p') \in M \times M'$, i.e. all r-jets of invertible local maps between M and M' with fixed source p and target p'.

Corollary 1.3. *Let M, M' and r be as in Theorem* 1.1. *Then, for any $p \in M$ and $p' \in M'$, there exist an open neighborhood Ω of $\{p\} \times G^r_{p,p'}(M, M')$ in $M \times G^r(M, M')$ and a smooth map $\Psi: \Omega \to M'$ such that every smooth CR-map f from any neighborhood of any point $q \in M$ into M' with $j^r_q f \in \Omega$ satisfies the identity*

$$f(x) = \Psi(x, j^r_q f) \tag{1.5}$$

for all $x \in M$ sufficiently close to q.

We remark that Corollary 1.3 gives a stronger statement than in [2,4,43,48,61] in the cases considered there. The main difference is that the parametrization in Corollary 1.3 is obtained simultaneously for all points q near a given one and for all local CR-maps in a neighborhood of q. The idea of the proof of Corollary 1.3 is to integrate the system (1.4) along curves in M chosen in a proper way. Vice versa, given a parametrization (1.5), the statement of Theorem 1.1 can be obtained by differentiating (1.5). For the latter conclusion, the mentioned stronger version of a parametrization with varying point q is essential.

Using Theorem 1.1 and Corollary 1.3 it is shown in [10] that under the assumptions of Theorem 1.1, the group $\mathrm{Aut}(M)$ of all (global) smooth (resp. real-analytic) CR-automorphisms of a smooth (resp. real-analytic manifold) M is a Lie group with respect to its natural C^∞ (resp. C^ω) topology. In contrast to $\mathrm{Aut}(M)$, the behaviour of the *local stability group* $\mathrm{Aut}(M, p)$ (that consists of germs at $p \in M$ of all local CR-automorphisms of M fixing p) is very different for smooth and for real-analytic CR-manifolds. In the real-analytic case, under the same assumptions as in Theorem 1.1, it was shown by the second named author [61] (see also [2] in the hypersurface case and more recent stronger results in [4]) that $\mathrm{Aut}(M, p)$ is a Lie group whose local action on M is real-analytic. In the smooth case $\mathrm{Aut}(M, p)$ has in general no Lie group structure (see [44]).

Corollary 1.3 can be used to obtain a solution to the CR equivalence problem in the following sense:

Theorem 1.4. *Let M, M' and r be as in Theorem* 1.1. *Then for any $p \in M$ there exist an open neighborhood Ω of $\{p\} \times G^r_p(M, M')$ in $M \times G^r(M, M')$ and a smooth real function $\Gamma\colon \Omega \to \mathbb{R}$ such that there exists a local smooth CR-diffeomorphism f between an open neighborhood of p in M and an open set in M' with $j^r_p f = \Lambda \in G^r_p(M, M')$ if and only if $\Gamma(\cdot, \Lambda)$ vanishes in a neighborhood of p.*

Finally we would like to mention that the condition of finite nondegeneracy in Theorem 1.1 cannot be replaced by essential finiteness (see e.g. [3] for the definition) as the following simple example shows.

Example 1.5. The hypersurface

$$M := \{(z, w) \in \mathbb{C}^2 : \operatorname{Im} w = |z|^4\}$$

is essentially finite and of finite type but not finitely nondegenerate at 0. We claim that the conclusion of Theorem 1.1 does not hold for M and $M' := M$. If it held, Corollary 1.3 would imply that the dimension of the local stability group $\operatorname{Aut}(M, q)$ is upper-semicontinuous with respect to $q \in M$. (Recall that $\operatorname{Aut}(M, q)$ consists of all germs at q of biholomorphic maps of $\mathbb{C}^2$ fixing q and sending M into itself.) On the other hand, M is locally biholomorphically equivalent to the quadric $Q := \{\operatorname{Im} w = |z|^2\}$ at any point (z, w) with $z \neq 0$ via the map $(z, w) \mapsto (z^2, w)$. Hence the group $\operatorname{Aut}(M, q)$ can be computed directly and it is easy to see that its dimension is not upper-semicontinuous which is a contradiction proving the above claim.

1.4. Complete systems for CR-embeddings

A fundamental problem naturally generalizing CR-equivalence problem is the following *CR-embedding problem*:

Given two manifolds with CR-structures, when does there exist a CR-embedding of one manifold into the other?

In our case the target manifold will always have positive CR-codimension, in particular, we do not consider here embeddings of CR-manifolds into complex manifolds. The CR-embedding problem stated here is related to the existence of proper holomorphic embeddings between smoothly bounded domains of different dimension the same way the CR-equivalence problem is related to the biholomorphic equivalence problem for such domains of the same dimension. Here the existence and regularity properties of embeddings are much less understood (see the surveys [33,40] for related results). Complete systems in some cases were constructed for real-analytic hypersurfaces due to [37,38,43].

In this paragraph we show how to extend the results from the previous paragraph to treat so-called finitely nondegenerate CR-embeddings between smooth (abstract) CR-manifolds (of any positive CR-codimension). We begin by adapting the finite nondegeneracy condition for an embedding i of an (abstract) CR-manifold M into an (abstract) CR-manifold M'. We shall identify M with its image $i(M)$ and hence see M as a submanifold of M'. Denote here by $T^{0,1}_M M'$ the sheaf of all $(0, 1)$ vector fields on M' that are tangent to M. (That is, a $(0, 1)$ vector field L on M' is in $\Gamma(T^{0,1}_M M')$ if and only if $L(x) \in T^{0,1}_x M$ for all $x \in M$ where L is defined.) The embedding $i\colon M \to M'$ is then called *finitely nondegenerate* at a point

$p \in M$ if, for some integer $k \geqslant 1$, in the notation of Section 1.2,

$$\begin{aligned}&\operatorname{span}_{\mathbb{C}}\{\mathscr{T}_{L_s}(\ldots\mathscr{T}_{L_2}(\mathscr{T}_{L_1}\theta)\ldots)(p) : 0 \leqslant s \leqslant k,\ \ L_j \in \Gamma(T_M^{0,1}M'),\\ &\quad \theta \in \Gamma(T^{*0}M')\} = T_p^{*1,0}M'.\end{aligned} \tag{1.6}$$

If l is the minimum of the integers k, the embedding is called *l-nondegenerate*. The arguments similar to those in [3, Section 11] show that, if M' is a CR-submanifold in $\mathbb{C}^{N'}$, the embedding of M is l-nondegenerate if and only if it is l-nondegenerate in the sense of [43,47]. It is easy to check that if an embedding of M into M' is finitely nondegenerate, then M' is itself finitely nondegenerate. The converse is not true: the simplest example is the standard linear embedding of the sphere $M = S^3 \subset \mathbb{C}^2$ into the sphere $M' = S^5 \subset \mathbb{C}^3$ which is not finitely nondegenerate.

In Theorem 1.1 we had a complete system defined for all invertible jets of smooth maps between M and M' whereas the nondegeneracy conditions were put on M and M'. In the case of embeddings we have the same finite type condition on M but the condition of finite nondegeneracy on M' is replaced by finite nondegeneracy of an embedding. In fact, the last condition involves only finitely many derivatives and hence can be considered as a condition on the jet of an embedding. We write here $D^{r,l}(M, M')$ for the subset of all r-jets of smooth embeddings that are CR and $\tilde{l}$-nondegenerate for some $\tilde{l} \leqslant l$. (The reader is referred to the next paragraph for a more precise definition and discussion of this notion.) We then have the following version of Theorem 1.1 for CR-embeddings:

Theorem 1.6. *Let M and M' be smooth (abstract) CR-manifolds, where M is of finite type. Then, for any $l \geqslant 0$ there exist a number $r > 0$, an open neighborhood Ω of $D^{r,l}(M, M')$ in $J^r(M, M')$ and a smooth map $\Phi\colon \Omega \to J^{r+1}(M, M')$ such that every smooth CR-embedding f of an open piece of M into M', which is $\tilde{l}$-nondegenerate at every point with $\tilde{l} \leqslant l$ depending on the point, satisfies the complete differential system*

$$j_x^{r+1} f = \Phi(j_x^r f) \tag{1.7}$$

for all x in the domain of definition of f. The number r can be chosen to be $2(d+1)l$, where d is the CR-codimension of M.

Theorem 1.6 is obtained as a special case of the more general statement given by Theorem 2.1 below. As a consequence, we obtain results analogous to those in Section 1.3 for finitely nondegenerate CR-embeddings.

A prototype of the above condition of l-nondegeneracy (in connection with a different problem) appeared in a paper of Cima et al. [26] and was later improved by Faran [31]. Faran gave his condition in different terms but it can be shown to be equivalent to that of 2-nondegeneracy in his case, where $M \subset \mathbb{C}^N$ and $M' \subset \mathbb{C}^{N'}$ are strongly pseudoconvex (or Levi-nondegenerate) real-analytic hypersurfaces. Later Han [37] gave a condition equivalent to that of l-nondegeneracy in this case for any l. In all these cases a certain partial normalization of the defining equation of M' has been used (that is known to exist due to the Chern–Moser normal form [23]). Note that no such normalization is known in general if M' is not a Levi-nondegenerate hypersurface or of higher codimension. On the other hand, the condition of l-nondegeneracy was further extended by the first author [43] to general real-analytic hypersurfaces and Lamel [46–48] to smooth embedded CR-manifolds of any codimension.

Remarkably, the condition of finite nondegeneracy for CR-embeddings that may look restrictive at first glance, holds sometimes automatically for all embeddings of manifolds of certain classes. We give two situations where this phenomenon occurs. The first situation was considered by Webster [59] in the case of a smooth CR-manifold M of hypersurface type embedded into the unit sphere $M' = S^{2n+1} \subset \mathbb{C}^{n+1}$ such that M has the real codimension 2 in M'. In this setting Webster established a fundamental relation between the pseudo-conformal curvature tensor S_M of M defined by Chern and Moser [23] and the second fundamental form of the embedding of M into M'. In fact, it follows from his formula [59, (2.14)] that, if S_M does not vanish at a point $p \in M$, then the restriction of the fundamental form to $T_p^c M$ also does not vanish for any embedding of M into M'. But the last condition means exactly 2-nondegeneracy of the embedding as was shown by Faran [31]. Hence we obtain the following application of Theorem 1.6:

Corollary 1.7. *Let $M' = S^{2n+1} \subset \mathbb{C}^{n+1}$ be the unit sphere and M be any smooth strongly pseudoconvex CR-manifold of hypersurface type of dimension $2n-1$ whose tensor S_M does not vanish at any point. Then the conclusion of Theorem* 1.6 *holds for all CR-embeddings of an open piece of M into M' with the jet order $r = 8$.*

Recall that S_M always vanish when M is 3-dimensional. On the other hand, in any higher dimension, if (M, p) is not locally equivalent to the sphere for some $p \in M$, then S_M does not identically vanish in a neighborhood of p.

We conclude by giving another general situation where all embeddings are automatically finitely nondegenerate. This time a condition is put only on the target manifold M'. We call a smooth CR-manifold M' *finitely nondegenerate in dimension n* at a point $p \in M'$ if, for any linearly independent $(0, 1)$ vector fields $X_1, \dots, X_n$ on M', there exists an integer k such that

$$\begin{aligned}&\operatorname{span}_{\mathbb{C}}\{\mathscr{T}_{L_s}(\dots\mathscr{T}_{L_2}(\mathscr{T}_{L_1}\theta)\dots)(p) : 0 \leqslant s \leqslant k,\ L_j \in \{X_1, \dots, X_n\},\\ &\quad \theta \in \Gamma(T^{*0}M)\} = T_p^{*1,0}M.\end{aligned} \tag{1.8}$$

The notion of l-nondegeneracy in dimension n is defined in the obvious way as before. It follows directly from the definition that if M' is finitely nondegenerate in dimension n at a point p' and if M has CR-dimension at least n then any embedding of M into M' through p' is finitely nondegenerate at this point. The following example shows that nondegenerate manifolds in the above sense exist already in the lowest dimension.

Example 1.8. We claim that the hypersurface $M' \subset \mathbb{C}^3$ given by

$$M' := \{(z_1, z_2, w) : \operatorname{Im} w = |z_1|^2 + |z_2|^2 + \operatorname{Im}(z_1^2\bar{z}_1 + (z_1 + z_2)^3(\bar{z}_1 + \bar{z}_2))\}$$

is 3-nondegenerate in dimension 1 at $p = 0$. Thus any embedding of any hypersurface $M \subset \mathbb{C}^2$ into M' through 0 is automatically l-nondegenerate at 0 for some $l \leqslant 3$. Indeed, given a $(0, 1)$ vector field X_1 with $X_1(0) \neq 0$, the span in (1.8) for $k = 1$ consists of all $(1, 0)$ forms vanishing on the orthogonal complement to $L_1(0)$ with respect to the Levi form $|z_1|^2 + |z_2|^2$. Furthermore, (1.8) holds for $k = 2$ unless $X_1(0) = (a, 0)$ or $X_1(0) = (0, b)$ for some $a, b \neq 0$. In the last case (1.8) holds for $k = 3$ which proves the claim.

We now have the following consequence of Theorem 1.6:

Theorem 1.9. *Let M and M' be smooth (abstract) CR-manifolds, M being of finite type and CR-dimension n and CR-codimension d and M' being l-nondegenerate in dimension n. Then the conclusion of Theorem* 1.6 *holds for all CR-embeddings of an open piece of M into M' with $r = 2(d+1)l$.*

2. Further results and generalizations

We now allow M and M' to be abstract CR-manifolds of possibly different CR-dimensions and CR-codimensions and consider CR-maps of finite smoothness that are not necessarily embeddings. If M is of class $\mathscr{C}^\kappa$, the condition on M to have the (finite) type ν at a point $p \in M$ makes sense for $\nu \leqslant \kappa$. The finite nondegeneracy condition from Section 1.4 extends to arbitrary CR-maps. As before, this condition will require a restricted number of differentiations and hence will make sense in the case where the map f and the manifolds M and M' have finite degree of smoothness. For an n-tuple of integers $\alpha = (\alpha_1, \dots, \alpha_n)$, we write $|\alpha| := \alpha_1 + \dots + \alpha_n$ and $L^\alpha := L_1^{\alpha_1} \dots L_n^{\alpha_n}$, where $L_1, \dots, L_n$ is a basis of $(0,1)$ vector fields near p. Suppose for a moment that M' is embedded in $\mathbb{C}^{N'}$ with a local defining function $\rho' = ({\rho'}^1, \dots, {\rho'}^{d'})$ near a point $p' \in M'$. A $\mathscr{C}^r$ CR-map f between M and M' with $f(p) = p'$ is said to be *l-nondegenerate* at p if

$$\operatorname{span}_{\mathbb{C}}\{L^\alpha {\rho'}^j_{Z'}(f(x), \overline{f(x)})(p) : 1 \leqslant j \leqslant d', 0 \leqslant |\alpha| \leqslant l\} = \mathbb{C}^{N'} \tag{2.1}$$

and if $l \geqslant 0$ is the minimal number for which (2.1) holds. Here ${\rho'}^j_{Z'} = (\partial {\rho'}^j / \partial Z'_1, \dots, \partial {\rho'}^j / \partial Z'_{N'})$ denotes the complex gradient with respect to the standard coordinates $(Z'_1, \dots, Z'_{N'}) \in \mathbb{C}^{N'}$ and hence ${\rho'}^j_{Z'}(f(x), \overline{f(x)})$ is a (vector-valued) differentiable function of $x \in M$. The definition makes sense for $l \leqslant \min(r, \kappa - 1)$. In this form it is due to [43,47,48]. The arguments in [47,48] show that this definition does not depend on the choice of the basis $L_1, \dots, L_n$, the defining function ρ' and the coordinates Z' (where the gradient ${\rho'}^j_{Z'}$ is computed). In particular, the given definition does not depend on the embedding of M into $\mathbb{C}^{N'}$.

An inspection of the above definition shows that it does not really use the whole CR-map f but rather its l-jet at p. Moreover, condition (2.1) can be written for any map $f: M \to M'$, not necessarily CR. It will be still independent of the choice of $L_1, \dots, L_n$ and of ρ' but will in general depend on the coordinates Z'. In order to eliminate this dependence, we restrict ourselves in this definition to the r-jets that we call *CR r-jets* and that are defined to be those r-jets at a point $p \in M$ of $\mathscr{C}^r$ maps $f: M \to M'$ that satisfy

$$f_* L(x) = o(|x - p|^{r-1}) \quad \mathrm{mod}\ T^{0,1}M' \ \text{ as } x \to p$$

for any $(0,1)$ vector field L on M. The latter condition means that $f_* L(x)$ for $x \in M$ near p can be written as a sum of two vectors $a(x)$ and $b(x)$ in $T_{f(x)}M' \otimes \mathbb{C}$ with $a(x) = o(|x-p|^{r-1})$ as $x \to p$ and $b(x) \in T^{0,1}_{f(x)}M'$ for all x. Any jet of a CR-map is a CR jet in this sense but, in general, there may be more CR r-jets (between M and M') at a given point p than CR-maps defined near p. We now call a CR r-jet $\Lambda \in J^r_p(M, M')$ *l-nondegenerate* ($l \leqslant r$) if (2.1) holds for some (and hence for any) representative f of Λ defined near p. Again, the arguments of [47,48] show that this definition is independent of the choice of $L_1, \dots, L_n$, ρ' and Z'.

We now extend the notion of l-nondegenerate maps to the case, where M' is not necessarily embedded in the same spirit as it was done in Section 1.2 for l-nondegenerate CR-manifolds and in Section 1.4 for

CR-embeddings. Since we wish to allow CR-maps that are not embeddings we cannot in general push (0, 1) vector fields from M to M'. The trick to avoid this difficulty is to consider the graph of f embedded into $M \times M'$. We call a $\mathscr{C}^r$ CR-map f as above *l-nondegenerate* at p if there exists an integer $0 \leqslant k \leqslant l$ such that

$$\begin{aligned} &\operatorname{span}_{\mathbb{C}}\{\mathscr{T}_{L_s}(\ldots\mathscr{T}_{L_2}(\mathscr{T}_{L_1}\theta)\ldots)(p, f(p)) : 0 \leqslant s \leqslant k,\ \ L_j \in \Gamma(T^{0,1}(M \times M')),\\ &\quad \theta \in \Gamma(T^{*0}(M \times M'))\} = T^{*1,0}_{(p,f(p))}(M \times M') \mod \pi^*(T^*M \otimes \mathbb{C}), \end{aligned} \tag{2.2}$$

where $\pi\colon M \times M' \to M$ denotes the standard projection, only those vector fields L_j are considered that are tangent to the graph of f, and $l \geqslant 0$ is the minimal number with this property. We call l the *degeneracy* of f at p. If f is defined on an open subset $U \subset M$, we call it l-nondegenerate if l is the maximal degeneracy of f at p for all $p \in U$. If M' is embedded, the given definition is equivalent to the one of Lamel [47]. On the other hand, if M' is merely abstract, it can be "approximately embedded" of sufficiently high order in the sense of Proposition 3.1 below and then l-nondegeneracy of CR-maps into M' can be equivalently defined through any such approximate embedding. Finally, the notions of CR r-jets and their l-nondegeneracy introduced above can be also directly extended to abstract CR-manifolds in an obvious way.

The main result stated in this section gives a complete differential system on the jet bundle $J^r(M, M')$ that is satisfied by all l-nondegenerate CR-maps between M and M'. We express it as a section of the fiber bundle $J^{r+1}(M, M') \to J^r(M, M')$ obtained by the natural projection of $(r+1)$-jets of maps from M to M' to their r-jets. As before, denote by $D^{r,l}(M, M') \subset J^r(M, M')$ the set of all CR r-jets in $J^r(M, M')$ that are $\tilde{l}$-nondegenerate for some $0 \leqslant \tilde{l} \leqslant l$.

Theorem 2.1. *Let M and M' be $\mathscr{C}^\kappa$ abstract CR-manifolds and denote by d the CR-codimension of M. Suppose that M is of the type ν, $2 \leqslant \nu \leqslant \kappa$, fix a number $l \geqslant 0$ and set*

$$r := 2(d+1)l, \quad k := 4(d^2+d)\nu l + 4(1-d^2)l + 2d\nu - 2d + 1. \tag{2.3}$$

Then, if $\kappa \geqslant k$, there exists an open neighborhood Ω of $D^{r,l}(M, M')$ in $J^r(M, M')$ and a $\mathscr{C}^{\kappa-k-1}$ section $\Phi\colon \Omega \to J^{r+1}(M, M')$ such that, for every open subset $U \subset M$ and every $0 \leqslant \tilde{l} \leqslant l$, every $\tilde{l}$-nondegenerate $\mathscr{C}^k$ CR-map $f\colon U \to M'$ satisfies

$$j^{r+1}_x f = \Phi(j^r_x f), \quad x \in U. \tag{2.4}$$

In the particular case when M is of hypersurface type (i.e. $d = 1$), (2.3) yields $k = 8\nu l + 2\nu - 1$. In general, a more refined estimate can be obtain by replacing ν with the sum of all Hörmander numbers of M at p due to Lemma 3.5 and Theorem 4.1 below.

Remark. It follows from the proof of Theorem 2.1 that the same conclusion holds also for every $x \in M$ and every l-nondegenerate $\mathscr{C}^k$ map f from a neighborhood of x in M into M' that is only CR up to order k at x rather than "precisely" CR.

Theorem 2.1 will be used to obtain a smooth local parametrization of CR-maps by their r-jets as follows:

Corollary 2.2. *With the notation and under the assumptions of Theorem 2.1, for every r-jet $\lambda_0 \in D^{r,l}(M, M')$ at a point $x_0 \in M$, there exists a $\mathscr{C}^{\kappa-k-1}$ map Θ from a neighborhood Ω of (x_0, λ_0) in*

$M \times J^k(M, M')$ into M' such that for every $p \in M$, $0 \leqslant \tilde{l} \leqslant l$ and an $\tilde{l}$-nondegenerate $\mathscr{C}^k$ CR-map f from a neighborhood $U(p) \subset M$ into M' with $(p, j_p^r f) \in \Omega$ satisfies the identity

$$f(x) \equiv \Theta(x, j_p^r f)$$

for all $x \in M$ in a neighborhood $\tilde{U}(p) \subset U(p)$ that depends on $U(p)$ and λ_0 but not on f.

An immediate consequence of Corollary 2.2 is the following regularity property for CR-maps:

Corollary 2.3. *With the notation and under the assumptions of Theorem* 2.1, *any l-nondegenerate $\mathscr{C}^k$ CR-map between open pieces of M and M' is necessarily of class $\mathscr{C}^{\kappa-k-1}$. In particular, if M and M' are $\mathscr{C}^\infty$, then f is also $\mathscr{C}^\infty$.*

For $\kappa = \infty$ and, when M and M' are embeddable, the statement of Corollary 2.3 was recently obtained by Lamel [46]. We would like to mention that classical methods for showing regularity as in Corollary 2.3 are based on holomorphic extension into wedges and hence cannot be directly applied to abstract nonembeddable CR-manifolds considered here.

3. Approximate Segre sets and iterated complexifications

3.1. Approximation of abstract CR-manifolds by embedded ones

We begin by proving a result on approximation of abstract CR-manifolds by embedded real-analytic ones. It is well-known (see e.g. [15]) that there exists a smooth abstract CR-manifold that is not locally embeddable in any $\mathbb{C}^N$. On the other hand, even if a CR-manifold M is smoothly embedded into $\mathbb{C}^N$, it can be shown that, in general, it may not be embedded as a real-analytic CR-submanifold (see e.g. [44]). Here we have a different point of view: we look for embeddings that are CR only up to some order. The following proposition shows that such embeddings always exist.

Proposition 3.1. *Let M be an abstract $\mathscr{C}^\kappa$ CR-manifold $(2 \leqslant \kappa \leqslant \infty)$ of CR-dimension n and CR-codimension d and let $k \leqslant \kappa$ be any integer. Then, for any $p \in M$, there exists a neighborhood U of p in M and a $\mathscr{C}^{\kappa-k}$ map $\varphi: U \times U \to \mathbb{C}^{n+d}$ such that for every $x \in U$, the following hold*:

(i) *$\varphi_x := \varphi(x, \cdot)$ is an embedding of U into $\mathbb{C}^{n+d}$ with $\varphi_x(x) = 0$ whose image is a generic real-analytic CR-submanifold of codimension d.*
(ii) *the map φ_x is CR of order k at x, i.e. $L\varphi_x(y) = o(|x - y|^{k-1})$ as $y \to x$ for any $(0, 1)$ vector field L on M defined in some neighborhood of x.*

We call $\varphi_x(U) \subset \mathbb{C}^N$ an *approximate* (*local*) *embedding of M* (*into $\mathbb{C}^N$*) *of order k at x*. For the proof of Proposition 3.1 we need the following approximate version of the holomorphic Frobenius theorem:

Lemma 3.2. *Let $k \geqslant 0$ be any integer and $L_1, \ldots, L_n$ be linearly independent holomorphic vector fields in $\mathbb{C}^m$ defined in a neighborhood of* 0 *such that*

$$[L_i, L_j](Z) = \sum c_{i,j}^s(Z) L_s(Z) + o(|Z|^k), \quad Z \to 0, \quad 1 \leqslant i, j \leqslant n, \tag{3.1}$$

for some holomorphic functions $c^s_{i,j}(Z)$. *Then there exist holomorphic vector fields* $\tilde{L}_1, \dots, \tilde{L}_n$ *spanning the same subspace as* $L_1, \dots, L_n$ *at every point and a local holomorphic change of coordinates* $\tilde{Z}$ *vanishing at* 0 *such that*

$$\tilde{L}_i = \frac{\partial}{\partial \tilde{Z}_i} + o(|\tilde{Z}|^{k+1}), \quad \tilde{Z} \to 0, \ \ 1 \leqslant i \leqslant n.$$

Moreover, if $L_1, \dots, L_n$ *depend smoothly* ($\mathscr{C}^k$) *on some given real parameters, the vector fields* $\tilde{L}_1, \dots, \tilde{L}_n$ *and the coordinates* $\tilde{Z}$ *can be chosen to depend smoothly on the same parameters with the same smoothness.*

Proof. Without loss of generality, $L_i(0) = \partial/\partial Z_i$ for $1 \leqslant i \leqslant n$. We write $Z = (z, w) \in \mathbb{C}^n \times \mathbb{C}^{m-n}$ for the local coordinates. Then we can further assume that $L_i(z, w) = \partial/\partial z_i + \sum a^j_i(z, w)(\partial/\partial w_j)$ for some holomorphic functions $a^j_i(z, w)$. For this choice of vector fields, (3.1) implies

$$[L_i, L_j](Z) = o(|Z|^k), \quad Z \to 0, \ \ 1 \leqslant i, j \leqslant n. \tag{3.2}$$

It is easy to see that there exists a holomorphic map χ from a neighborhood of 0 in $\mathbb{C}^m$ into $\mathbb{C}^{m-n}$ such that

$$(\partial^\alpha_z \chi^i)(0, w) = (L^\alpha w^i)(0, w), \quad 0 \leqslant |\alpha| \leqslant k+2, \ \ 1 \leqslant i \leqslant m-n, \tag{3.3}$$

where $\alpha \in \mathbb{Z}^n_+$ is any multiindex. Furthermore, the identity (3.2) allows us to permute the order of applying the vector fields on the right-hand side of (3.3) with a controlled error term:

$$(\partial_{z_{i_1}} \dots \partial_{z_{i_s}} \chi^i)(0, w) = (L_{i_1} \dots L_{i_s} w^i)(0, w) + o(|w|^{k+2-s}), \tag{3.4}$$

where $i_1, \dots, i_s \in \{1, \dots, n\}$, $0 \leqslant s \leqslant k+2$, $1 \leqslant i \leqslant m-n$. Then it follows that the change of coordinates given by $\tilde{Z}(z, w) = (z, \chi(z, w))$ satisfies the required conclusion. The last statement about the dependence on parameters follows directly from the proof. □

Proof of Proposition 3.1. The proof can be obtained by following the arguments in the proof of Theorem 2.1.11 in [3] and using approximate complexifications of the smooth vector fields and Lemma 3.2 instead of Theorem 2.1.12 in [3]. The details are left to the reader. □

The constructive proof of Proposition 3.1 can be directly extended to yield the following parameter version of Proposition 3.1 that we state here, for simplicity, only in the smooth ($\mathscr{C}^\infty$) case:

Proposition 3.3. *Let* M *be a smooth manifold,* $V \subset \mathbb{R}^s$ *an open set and let on every* $M \times \{v\}$, $v \in V$, *be given an* (*abstract*) *smooth CR structure of CR-dimension* n *and CR-codimension* d *which depends smoothly on* $(p, v) \in M \times V$. *Then, given an integer* k *and a point* $(p_0, v_0) \in M \times V$, *there exist a neighborhood* $U \times V_0$ *of* (p_0, v_0) *and a smooth map* $\tilde{\varphi}: U \times U \times V_0 \to \mathbb{C}^{n+d}$ *such that, for every* $v \in V_0$, *the map* $\varphi := \tilde{\varphi}(\cdot, \cdot, v): U \times U \to \mathbb{C}^{n+d}$ *satisfies both properties* (i) *and* (ii) *in Proposition* 3.1.

3.2. Complexifications of smooth families of generic manifolds

In the sequel we use the semicolon ";" to separate the variables, where the functions (or maps) are holomorphic, from the other variables, where they are merely continuously differentiable of certain order.

Motivated by Proposition 3.1 we consider a general $\mathscr{C}^\kappa$ map $\varphi: U \times \tilde{U} \to \mathbb{C}^N$ with $U \times \tilde{U} \subset \mathbb{R}^{2n+d} \times \mathbb{R}^h$ (h is a fixed integer) being an open neighborhood of the origin such that every $\varphi_x := \varphi(\cdot, x)$ is an embedding of U into $\mathbb{C}^N$ with $\varphi_x(U) \ni 0$ and whose image is a real-analytic generic submanifold of CR-dimension n and CR-codimension d. As before $N = n + d$. We write $M(x) := \varphi_x(U)$.

By the implicit function theorem, there exists a $\mathscr{C}^\kappa$ family of local ($\mathbb{C}^d$-valued) defining functions of $M(x)$, $\rho(Z, \overline{Z}; x)$, defined in a neighborhood of 0, that are holomorphic in the first two arguments and vanish at $Z=0$ for every x. As usual, we write $\rho=(\rho^1, \dots, \rho^d)$ and assume $d\rho^1(\cdot, \cdot; x) \wedge \cdots \wedge d\rho^d(\cdot, \cdot; x) \neq 0$ on the domain of definition for each fixed x. Finally, we can choose a neighborhood, where $\rho(\cdot, \cdot; x)$ is defined, to be independent of x and k (each $\rho(\cdot, \cdot; x)$ can be chosen to be a polynomial of degree k).

We can now define a family of *complexifications of* $M(x)$ by

$$\mathscr{M}(x) := \{(Z, \zeta) : \rho(Z, \zeta; x) = 0\}.$$

Then $\mathscr{M}(x)$ is a complex submanifold of $\mathbb{C}^N \times \mathbb{C}^N$ through 0 of complex codimension d. We further consider (cf. [1,4,11,61]), for every $s \geqslant 1$, the family

$$\mathscr{M}^s(x) := \{\xi = (\xi^0, \dots, \xi^s) \in \mathbb{C}^{(s+1)N} : \xi^s = 0, \rho_j(\xi^{j-1}, \xi^j; x) = 0 \text{ for all } j = 1, \dots, s\},$$

where

$$\rho_j(\xi^{j-1}, \xi^j; x) := \begin{cases} \overline{\rho}(\xi^{j-1}, \xi^j; x) & \text{if } j \text{ is even,} \\ \rho(\xi^{j-1}, \xi^j; x) & \text{if } j \text{ is odd.} \end{cases}$$

(We write the index of ρ below to avoid the confusion with the component indices that we wrote above.) It follows that, for every $x \in M$ (close to p), $\mathscr{M}^s(x) \subset \mathbb{C}^{(s+1)N}$ is a complex submanifold of dimension $sn + N$ through the origin.

3.3. *Finite type for smooth abstract CR-manifolds*

Let M be an abstract $\mathscr{C}^\kappa$ CR-manifold having finite type v at a point $p \in M$ with $1 \leqslant v \leqslant k \leqslant \kappa$. Then the approximation $M(p) = \varphi_p(U)$ given by Proposition 3.1 is also of finite type v at $0 \in M(p)$.

Going back to a general family $M(x) \subset \mathbb{C}^N$, $x \in \tilde{U} \subset \mathbb{R}^h$, as in the previous section, assume that $p=0$ and $M(0)$ is of finite type v at $0 \in M(0)$. After a possible biholomorphic change of local coordinates $Z = (z, w) \in \mathbb{C}^n \times \mathbb{C}^d$ in a neighborhood of the origin we may assume that $M(x)$'s are locally given by

$$w = Q(z, \overline{z}, \overline{w}; x), \tag{3.5}$$

where $Q(\cdot, \cdot, \cdot; x)$, $x \in \tilde{U}$, is a $\mathscr{C}^\kappa$ family of holomorphic maps from a neighborhood of 0 in $\mathbb{C}^n \times \mathbb{C}^d$ into $\mathbb{C}^d$, vanishing at 0. In addition we may assume that the coordinates are normal for $M(0)$ (see e.g. [3], Section 4.2), i.e.

$$Q(z, 0, \tau; 0) \equiv Q(0, \chi, \tau; 0) \equiv \tau.$$

We next consider local parametrizations (cf. [11, Section 10])

$$\mathbb{C}^{2sn} \times \tilde{U} \ni (t^0, \dots, t^{2s-1}; x) \mapsto \Xi^s(t^0, \dots, t^{2s-1}; u) \in \mathscr{M}^{2s} \subset \mathbb{C}^{(2s+1)N} \times \tilde{U} \tag{3.6}$$

near the origin, where

$$\Xi^s(t^0,\dots,t^{2s-1};x) := (v^{2s}(t^0,\dots,t^{2s-1};x), \overline{v^{2s-1}}(t^1,\dots,t^{2s-1};x),\dots,v^2(t^{2s-2},t^{2s-1};x), \overline{v^1}(t^{2s-1};x),0;x) \tag{3.7}$$

and the components

$$v^s\colon (\mathbb{C}^{sn}\times\tilde U,0)\to(\mathbb{C}^N,0),\quad s=0,1,\dots,$$

are defined (as germs at 0) inductively as follows. Set $v^0(x):=0$ and

$$v^{s+1}(t^0,t^1,\dots,t^s;x) := (t^0, Q(t^0,\overline{v^s}(t^1,\dots,t^s;x)))\in\mathbb{C}^n\times\mathbb{C}^d \tag{3.8}$$

for $s\geqslant 0$, $t^0,t^1,\dots,t^s\in\mathbb{C}^n$ and $x\in\tilde U$. It is easy to check that for $s\geqslant 0$,

$$(v^{s+1}(t^0,t^1,\dots,t^s;x),\overline{v^s}(t^1,\dots,t^s;x))\in\mathscr{M}(x), \tag{3.9}$$

and hence (3.7) defines a family of parametrizations of $\mathscr{M}(x)$ as desired. By the construction, Ξ^s is a function of class $\mathscr{C}^\kappa$ in its variables and holomorphic in $(t^0,\dots,t^{2s-1})$ for x fixed.

We now observe that, since the coordinates in $\mathbb{C}^N=\mathbb{C}^n\times\mathbb{C}^d$ are chosen to be *normal* for $M(0)$ at 0, $v^s(\cdot;0)$ coincides with the sth *Segre map* (or its conjugate) in the sense of [4]. Hence by [4, Theorem 3.1.9], the generic rank of $v^s(\cdot;0)$ equals N for $s\geqslant d+1$.

By Lemma 4.1.3 in [4], we have

$$v^{2s}(0,u^1,\dots,u^{s-1},u^s,u^{s-1},\dots,u^1;x)\equiv 0,$$

(this identity is stated in [4] only for normal coordinates but can be obtained using the same argument in general case) and the generic rank of the partial derivative matrix

$$\frac{\partial v^{2s}}{\partial(t^0,t^{s+1},t^{s+2},\dots,t^{2s-1})}$$

along the linear subspace

$$\{(0,u^1,\dots,u^{s-1},u^s,u^{s-1},\dots,u^1;0):(u^1,\dots,u^{s-1},u^s)\in\mathbb{C}^{sn}\}$$

equals N. As in [4] we introduce variables $\eta=(\eta^1,\dots,\eta^s)\in\mathbb{C}^{sn}$ and $\xi=(\xi^0,\dots,\xi^{s-1})\in\mathbb{C}^{sn}$ and put

$$\tilde V(\eta,\xi;x) := v^{2s}(\xi^0,\eta^1,\dots,\eta^{s-1},\eta^s,\eta^{s-1}+\xi^{s-1},\eta^{s-2}+\xi^{s-2},\dots,\eta^1+\xi^1;x). \tag{3.10}$$

Then the generic rank of $\partial\tilde V/\partial\xi$ along $\{(\eta,0;x)\}$ equals N for $x\in\tilde U$ near 0 and we can shrink $\tilde U$ around 0, reorder the sn components of ξ, and write $\xi=(\xi^1,\xi^2)\in\mathbb{C}^N\times\mathbb{C}^{sn-N}$ such that the determinant of $(\partial\tilde V/\partial\xi^1)(\eta,0;x)$ does not vanish identically for each $x\in\tilde U$. Hence the map

$$V(\eta,\xi^1;x) := \tilde V(\eta,(\xi^1,0);x) \tag{3.11}$$

satisfies the assumptions of the following lemma (where ξ^1 is replaced by ξ for brevity). This lemma is a variant of Proposition 4.1.18 in [4] with smooth parameters (see also Lemma 10.5 in [11]). The proof given here closely follows the arguments of [4].

Lemma 3.4. *Let $V(\eta, \xi; x)$ be a map of class $\mathscr{C}^\kappa$ from a neighborhood of* 0 *in $\mathbb{C}^h \times \mathbb{C}^N \times \mathbb{R}^m$ into $\mathbb{C}^N$ with $V(\cdot, \cdot; x)$ holomorphic for each fixed x. Suppose that $V(\eta, 0; x) \equiv 0$ and*

$$\delta(\eta; x) := \det\left(\frac{\partial V}{\partial \xi}(\eta, 0; x)\right)^2 \not\equiv 0 \tag{3.12}$$

for each fixed x. Then there exists a map $\varphi(\eta, \tilde{Z}; x)$ from a neighborhood of 0 *in $\mathbb{C}^h \times \mathbb{C}^N \times \mathbb{R}^m$ into $\mathbb{C}^N$ of class $\mathscr{C}^\kappa$ and holomorphic for each fixed x, such that the identity*

$$V\left(\eta, \varphi\left(\eta, \frac{Z}{\delta(\eta; x)}; x\right); x\right) \equiv Z$$

holds for all $(\eta, Z; x)$ with $(\eta; x)$ and $Z/\delta(\eta; x)$ sufficiently small.

Proof. Since $V(\eta, \xi; x)$ is holomorphic in (η, ξ) and $V(\eta, 0; x) \equiv 0$, we can write it in the form

$$V(\eta, \xi; x) = V_1(\eta, \xi; x)\xi,$$

where V_1 is the $N \times N$ matrix function of class $\mathscr{C}^\kappa$ given by

$$V_1(\eta, \xi; x) := \int_0^1 \frac{\partial V}{\partial \xi}(\eta, t\xi; x)\,\mathrm{d}t.$$

In particular, we have

$$V_1(\eta, 0; x) = \frac{\partial V}{\partial \xi}(\eta, 0; x). \tag{3.13}$$

The map V_1 is also of class $\mathscr{C}^\kappa$ because of the Cauchy type formula for the derivatives. By applying the same procedure one more time we can write $V_1(\eta, \xi; x) = V_1(\eta, 0; x) + V_2(\eta, \xi; x)\xi$ with V_2 again being of class $\mathscr{C}^\kappa$. Then

$$V(\eta, \xi; x) = V_1(\eta, 0; x)\xi + R(\eta, \xi; x)(\xi, \xi),$$

where $R(\eta, \xi; x)(\cdot, \cdot)$ is the bilinear form defined by V_2.

By (3.12) and (3.13) we have $\det(V_1(\eta, 0; x)) \not\equiv 0$ for each fixed x. We set $\Delta(\eta; x) := \det(V_1(\eta, 0; x))$. We now proceed exactly as in the proof of Proposition 4.1.18 in [4]. We write the equation $V(z, \xi; x) = Z$ as

$$V_1(\eta, 0; x)\xi + R(\eta, \xi; x)(\xi, \xi) = Z$$

and solve it for ξ by using Cramer's rule in terms of $Z/\Delta(\eta; x)$ and $R(\eta, \xi; x)(\xi/\Delta(\eta; x), \xi)$. In the expression obtained we divide both sides by $\Delta(\eta; x)$ and apply the implicit function theorem with respect to $\xi/\Delta(\eta; x)$:

$$\frac{\xi}{\Delta(\eta; x)} = \psi\left(\eta, \frac{Z}{\Delta(\eta; x)^2}; x\right), \quad \psi \in \mathscr{C}^\kappa, \ \ \psi(\cdot, \cdot; x) \text{ holomorphic.}$$

Then the map

$$\varphi(\eta, \tilde{Z}; x) := \Delta(\eta; x)\psi(\eta, \tilde{Z}; x) \in \mathbb{C}^N$$

satisfies the conclusion of the lemma. □

3.4. Hörmander numbers and the vanishing order of $\delta(\eta; x)$

As in the previous section, let us begin with an abstract $\mathscr{C}^\kappa$ CR-manifold M which is of finite type $\nu \leqslant \kappa$ at $p \in M$. Recall that the *Hörmander numbers* $2 \leqslant \mu_1 \leqslant \cdots \leqslant \mu_d = \nu$ *of* M *at* p are defined as follows. For $1 \leqslant j \leqslant d$, the jth Hörmander number μ_j is the minimal integer $\mu \leqslant \nu$ with the property that all vector fields in $T^{\mathrm{c}}M$ (of class $\mathscr{C}^{\kappa-1}$) together with their commutators of length less or equal to μ span at p a vector subspace of T_pM of dimension at least $n + j$. In particular, we have $\mu_d = \nu$. (This definition differs slightly from the one given in Section 3.4 in [3], where the same Hörmander numbers are counted as a single number with multiplicity.)

If φ is given by Proposition 3.1 for a fixed k with $\nu \leqslant k \leqslant \kappa$, property (ii) there implies that also $M(p) = \varphi_p(U)$ has finite type ν at 0. Furthermore, since the values at p of all commutators of vector fields in $T^{\mathrm{c}}M(p)$ of length at most k coincide with those of vector fields in $T^{\mathrm{c}}M$, all Hörmander numbers of $M(p)$ at p coincide with those of M.

Again, going back to a general smooth family $M(x)$ as before, assume that $M(0)$ is of finite type ν at 0 and has Hörmander numbers $2 \leqslant \mu_1 \leqslant \cdots \leqslant \mu_d = \nu$ at 0. We now use the so-called *normal canonical coordinates* (see [9,14]). For our purposes, it will be convenient to have the normal canonical coordinates in the form of Theorem 3.2.3 in [4] (see also Theorem 4.5.1 in [3]). It follows that there exist holomorphic local coordinates $Z = (z, w) \in \mathbb{C}^n \times \mathbb{C}^d = \mathbb{C}^N$ in a neighborhood of p, where $M(0)$ is locally given by the equations

$$w_j = \overline{w}_j + +P_j(z, \overline{z}, \overline{w}) + R_j(z, \overline{z}, \overline{w}), \quad 1 \leqslant j \leqslant d,$$

where the functions P_j and R_j are holomorphic in their arguments, satisfy the normalization conditions

$$P_j(z, 0, \overline{w}) \equiv R_j(z, 0, \overline{w}) \equiv P_j(0, \overline{z}, \overline{w}) \equiv R_j(0, \overline{z}, \overline{w}) \equiv 0$$

and, in the notation $\varepsilon^\mu w := (\varepsilon^{\mu_1} w_1, \ldots, \varepsilon^{\mu_d} w_d)$, the weight homogeneity conditions

$$P_j(\varepsilon z, \varepsilon\overline{z}, \varepsilon^\mu \overline{w}) = \varepsilon^{\mu_j} P_j(z, \overline{z}, \overline{w}), \quad R_j(\varepsilon z, \varepsilon\overline{z}, \varepsilon^\mu \overline{w}) = O(\varepsilon^{\mu_j+1})$$

for small $\varepsilon > 0$. The functions $P_j(z, \overline{z}, \overline{w})$ are nonzero polynomials and the submanifold $M_0 := \{w_j = \overline{w}_j + P_j(z, \overline{z}, \overline{w}), 1 \leqslant j \leqslant d\}$ is of the same finite type ν and has the same Hörmander numbers (M_0 is a *weighted homogeneous generic submanifold* in the sense of Section 4.4 in [3]).

As was observed in [4], in the canonical coordinates introduced above, the map $v^s(\cdot, \cdot; p)$ has a power series expansion whose first n components are linear and, for $1 \leqslant j \leqslant d$, its $(n + j)$th component starts from a nonzero homogeneous polynomial of degree μ_j. The same holds for the components of V (defined in (3.11)). Hence the power series expansion of the determinant in (3.12) starts from a homogeneous polynomial of degree $(\mu_1 - 1) + \cdots + (\mu_d - 1) = \mu_1 + \cdots + \mu_d - d$. Since M_0 is of finite type at p, the determinant cannot vanish identically. Thus we come to the following conclusion:

Lemma 3.5. *Let* $V(\eta, \xi; x)$ *be given by* (3.11) *and* $\delta(\eta; x)$ *be given by Lemma* 3.4. *Then there exists a vector* $\eta_0 \in \mathbb{C}^{sn}$ *such that the vanishing order* m *of the function* $\tilde{\delta}(\lambda) := \delta(\lambda\eta_0; p)$ *at* $\lambda = 0 \in \mathbb{C}$ *equals* $2(\mu_1 + \cdots + \mu_d - d)$. *In particular,* $m \leqslant 2d(\nu - 1)$.

4. Jet parametrization of local holomorphic maps

Consider two generic CR-submanifolds $M \subset \mathbb{C}^N$ and $M' \subset \mathbb{C}^{N'}$. If $p \in M$ and $p' \in M'$ are fixed and $f: (M, p) \to (M', p')$ is a germ of a $\mathscr{C}^r$ CR-map (or, more generally, of a $\mathscr{C}^r$ map which is CR up to order r), it can be shown (see e.g. the arguments in the proof of Proposition 1.7.14 in [3]) that f can be approximated up to order r by the restriction to M of a holomorphic polynomial map $F: \mathbb{C}^N \to \mathbb{C}^{N'}$ of degree r satisfying

$$\rho'(F(x), \overline{F(x)}) = o(|x - p|^r) \quad \text{as } x \to p \text{ in } M, \tag{4.1}$$

where $\rho' = ({\rho'}^1, \dots, {\rho'}^{d'})$ is a local defining function for M' vanishing at 0 with $\partial {\rho'}^1 \wedge \cdots \wedge \partial {\rho'}^{d'} \neq 0$ on M'. If (4.1) holds, we say that F *sends M into M' up to order r at p*. This notion extends that of $(r+1)$-equivalence given in [11]. If M is generic at p, i.e. if $T_pM + JT_pM = T_p\mathbb{C}^N$ for any $p \in M$, then the jet $j^r_p F \in J^r(\mathbb{C}^N, \mathbb{C}^{N'})$ is uniquely determined by $j^r_p f \in J^r(M, M')$. We also say that an r-jet $\Lambda \subset J^r_p(\mathbb{C}^N, \mathbb{C}^{N'})$ *sends M into M'* if some (and hence any) holomorphic representative F of Λ sends M to M' up to order r at p. The condition (2.1) of l-nondegeneracy can be also written with f replaced by F and is again independent of the choice of $L_1, \dots, L_n$ and of Z' provided F is holomorphic. If, in addition, F sends$M$ into M' up to order $r \geqslant l+1$ at x, then this condition is also independent of the choice of ρ'. In this case we call both F and its jet $j^r_p F$ *l-nondegenerate* (*with respect to M and M'*). Following the line of the notation of Section 2, we denote by $D^{r,l}_{M,M'} \subset J^r(\mathbb{C}^N, \mathbb{C}^{N'})$ the subset of all jets that send M into M' and are $\tilde{l}$-nondegenerate for some $\tilde{l} \leqslant l$ with respect to M and M'. For $x \in M$ and $x' \in M'$ we also write $D^{r,l}_{M,x;M',x'} := D^{r,l}_{M,M'} \cap J^r_{x,x'}(\mathbb{C}^N, \mathbb{C}^{N'})$ for the corresponding subset of jets with fixed source x and target x'.

In the previous section we have shown how to approximate an abstract smooth CR-manifold M by a smooth family of embedded real-analytic generic submanifolds of $\mathbb{C}^{n+d}$ passing through 0, where n and d are respectively CR-dimension and CR-codimension of M. In this section we consider a general $\mathscr{C}^\kappa$ (resp. $\mathscr{C}^\omega$) family of generic submanifolds $M(x) \subset \mathbb{C}^N$ of codimension d through 0 (with $N := n + d$), defined for $x \in \mathbb{R}^s$ near 0 (for some fixed integer $s \geqslant 0$), in the sense that there exists a $\mathscr{C}^\kappa$ (resp. $\mathscr{C}^\omega$) simultaneous parametrization

$$\varphi: U \times V \subset \mathbb{R}^{2n+d} \times \mathbb{R}^s \to \mathbb{C}^N,$$

where $U \times V$ is an open neighborhood of the origin, such that $\varphi_x := \varphi(\cdot; x): U \to \mathbb{C}^N$ is an embedding with image $M(x) = \varphi_x(U)$ for every $x \in V$. We use this terminology in the following statement which is the main technical result of this section:

Theorem 4.1. *Let $U \times U' \subset \mathbb{R}^s \times \mathbb{R}^{s'}$ be a neighborhood of the origin and $M(x) \subset \mathbb{C}^N$, $M'(x') \subset \mathbb{C}^{N'}$, $(x, x') \in U \times U'$, be $\mathscr{C}^\kappa$ $(2 \leqslant \kappa \leqslant \infty)$ families of generic real-analytic submanifolds passing through the origin. Suppose that $M(0)$ is of codimension $d \geqslant 1$ and of finite type $2 \leqslant \nu \leqslant \kappa$. Fix a number $l \geqslant 0$, set $r := 2(d+1)l$ and fix a jet $\Lambda_0 \in D^{r,l}_{M(0),0;M'(0),0}$. Then there exist a neighborhood Ω of $(0, \Lambda_0, 0, 0)$ in $\mathbb{C}^N \times J^r(\mathbb{C}^N, \mathbb{C}^{N'}) \times U \times U'$ and, for every k with $\max(r, \nu) \leqslant k \leqslant \kappa - 1$, a $\mathscr{C}^{\kappa-1}$ map $\Psi^k(Z, \Lambda; x, x'): \Omega \to \mathbb{C}^{N'}$, holomorphic in $(Z, \Lambda) \in \mathbb{C}^N \times J^r_{0,0}(\mathbb{C}^N, \mathbb{C}^{N'})$ for each fixed (x, x'), such that the following holds. For every $(x, x') \in U \times U'$, every holomorphic map F from a neighborhood of 0 in $\mathbb{C}^N$ into $\mathbb{C}^{N'}$ with*

$F(0)=0$ and $(0, j_0^r F; x, x') \in \Omega$ that sends $M(x)$ into $M'(x')$ up to order k at 0, satisfies the identity

$$F(Z) = \Psi^k(Z, j_0^r F; x, x') + o\left(|Z|^{\frac{k-r}{m+1}}\right), \quad as\ Z \to 0, \tag{4.2}$$

where $m \leqslant 2d(\nu-1)$ is an integer. If $2 \leqslant \mu_1 \leqslant \cdots \leqslant \mu_d = \nu$ are Hörmander numbers of $M(0)$ at 0, we can set $m := 2(\mu_1 + \cdots + \mu_d - d)$. If the families $M(x)$ and $M'(x')$ are in addition $\mathscr{C}^\omega$, the maps Ψ^k can also be chosen $\mathscr{C}^\omega$.

The rest of this section is devoted to the proof of Theorem 4.1 that is split into four subsections.

4.1. Basic reflection identity

Our first goal will be to establish a relation between jets of a holomorphic map F as in Theorem 4.1 at two different points Z and ζ satisfying the reflection relation $(Z, \zeta) \in \mathscr{M}(x)$, where $\mathscr{M}(x)$ is the complexification of $M(x)$ as in Section 3.2. For $x \in U$ near 0, fix a $\mathscr{C}^\kappa$ family of real-analytic defining functions $\rho(Z, \overline{Z}; x)$ of $M(x)$. We further choose a $\mathscr{C}^\kappa$ family of holomorphic functions $\rho(Z, \overline{Z}; x)$ as in Section 3.2. Then it follows form Cauchy type formulas for the derivatives of holomorphic functions that partial derivatives of $\rho(Z, \overline{Z}; x)$ in $(Z, \overline{Z})$ are also $\mathscr{C}^\kappa$ in all variables. Hence we can fix a basis $L_j(x) = L_j(Z, \overline{Z}; x)$ of $(1,0)$ vector fields in $\mathbb{C}^N$ near 0 for $x \in U$ near 0, which are real-analytic in $(Z, \overline{Z})$ and $\mathscr{C}^\kappa$ in all their variables.

As in Theorem 4.1, consider an l-nondegenerate r-jet $\Lambda_0 \in D^{r,l}_{M(0),0;M'(0),0}$, a point $(x, x') \in U \times U'$ near 0 and a holomorphic map F from a neighborhood of 0 in $\mathbb{C}^N$ into $\mathbb{C}^{N'}$ sending 0 to 0 and sending $M(x)$ to $M'(x')$ up to order k at 0. We shall assume (x, x') and the r-jet $j_0^r F$ to be sufficiently close to $(0,0)$ and Λ_0 respectively. Then, for $f := F|_{M(x)}$, the condition (2.1) still holds with p replaced by 0 and ρ' by $\rho'(\cdot, \cdot; x')$. We conclude that

$$\operatorname{span}\{L(x)^\alpha {\rho'}^j_{Z'}(F(Z), \overline{F(Z)}; x')(0) : 1 \leqslant j \leqslant d', 0 \leqslant |\alpha| \leqslant l\} = \mathbb{C}^{N'} \tag{4.3}$$

for F as above.

Since F sends $M(x)$ into $M'(x')$ up to order k at 0, we have

$$\rho'(F(Z), \overline{F(Z)}; x') = o(|Z|^k) \quad \text{as } Z \to 0 \text{ in } M(x). \tag{4.4}$$

By applying vector fields $L_j(x)$ to (4.4) we obtain

$$L(x)^\alpha \rho'(F(Z), \overline{F(Z)}; x') = o(|Z|^{k-|\alpha|}) \quad \text{as } Z \to 0 \text{ in } M(x), \quad |\alpha| \leqslant l. \tag{4.5}$$

Since each $L_j(Z, \overline{Z}; x)$ is real-analytic in $(Z, \overline{Z})$, we can consider the complexifications $L_j(Z, \zeta; x)$. By the standard complexification argument, (4.5) implies

$$L(Z, \zeta; x)^\alpha \rho'(F(Z), \overline{F}(\zeta); x') = o(|(Z, \zeta)|^{k-|\alpha|}) \quad \text{as } (Z, \zeta) \to 0 \text{ in } \mathscr{M}(x), \tag{4.6}$$

where we use the notation from previous sections. Recall that $\rho'(Z', \zeta'; x')$ as well as each $L(Z, \zeta; x)$ is of class $\mathscr{C}^\kappa$ and holomorphic in (Z', ζ') and in (Z, ζ) respectively. Under the assumptions of Theorem 4.1 we clearly have $l \leqslant k$. Since all differentiations in (4.6) are taken in the "holomorphic directions", we conclude that the left-hand side of (4.6) is of class $\mathscr{C}^\kappa$ (see similar arguments in the proof of Lemma 3.4). By using the chain rule and the fact that $L(Z, \zeta; x)^\alpha \overline{F}(\zeta) \equiv 0$, we can write the left-hand side of (4.6)

for $|\alpha|\leqslant l$ as a $\mathscr{C}^\kappa$ function in $(Z, j_Z^l F, \zeta, \overline{F}(\zeta); x, x')$, holomorphic in $(Z, j_Z^l F, \zeta, \overline{F}(\zeta))$ for each fixed (x, x') near the origin.

In view of (4.3), the implicit function theorem can be used to solve (4.6) for $\overline{F}(\zeta)$ in the form

$$\overline{F}(\zeta) = \Psi^0(\zeta, Z, j_Z^l F; x, x') + o(|(Z,\zeta)|^{k-l}) \quad \text{as } (Z,\zeta)\to 0 \text{ in } \mathscr{M}(x), \tag{4.7}$$

where $\Psi^0(\zeta, Z, \Lambda; x, x')$ is a $\mathbb{C}^{N'}$-valued function of class $\mathscr{C}^\kappa$ in a neighborhood of $(0, 0, \Lambda_0; 0, 0)$, holomorphic in (ζ, Z, Λ) for each fixed (x, x').

We next differentiate (4.7) in Z. Since (4.7) holds for $(Z,\zeta)\in\mathscr{M}(x)$, we first have to express ζ as a function of Z. Due to the construction of $\mathscr{M}(x)$, for $((Z_0,\zeta_0); x)\in\mathscr{M}(x)\times U$ close to the origin, there exists a $\mathscr{C}^\kappa$ function $Z = Z(\zeta, Z_0, \zeta_0; x)\in\mathbb{C}^N$ in a neighborhood of 0, with $Z(\zeta_0, Z_0, \zeta_0; x)\equiv Z_0$, holomorphic in (ζ, Z_0, ζ_0) and satisfying $(Z(\zeta, Z_0, \zeta_0; x), \zeta)\in\mathscr{M}(x)$. After substituting $Z(\zeta, Z_0, \zeta_0; x)$ for Z in (4.7) and differentiating in ζ we obtain the *basic reflection identity*

$$j_\zeta^\tau \overline{F} = \Psi^\tau(\zeta, Z, j_Z^{\tau+l} F; x, x') + o(|(Z,\zeta)|^{k-l-\tau}) \quad \text{as } (Z,\zeta)\to 0 \text{ in } \mathscr{M}(x), \tag{4.8}$$

$\tau\leqslant k-l$, where $\Psi^\tau(\zeta, Z, \Lambda; x, x')$ is still of class $\mathscr{C}^\kappa$, because we have differentiated only in the "holomorphic directions".

Recall that the r-jet $\Lambda_0\in J_{0,0}^r(\mathbb{C}^N,\mathbb{C}^{N'})$ is assumed to send M into M'. Then, by following the construction, we see that (4.8) also holds with k replaced by r and any representative F of Λ_0. Hence we obtain

$$\overline{\Lambda_0^\tau} = \Psi^\tau(0, 0, \Lambda_0^{\tau+l}; 0, 0)\in J_{0,0}^\tau(\mathbb{C}^N,\mathbb{C}^{N'}) \tag{4.9}$$

for $\tau+l\leqslant r$, where $\Lambda_0^\tau\in J_{0,0}^\tau(\mathbb{C}^N,\mathbb{C}^{N'})$ denotes the image of Λ_0 under the canonical projection $J^r(\mathbb{C}^N,\mathbb{C}^{N'})\to J^\tau(\mathbb{C}^N,\mathbb{C}^{N'})$.

We summarize the result of this paragraph indicating the dependence of Ψ^τ on k by adding a superscript:

Proposition 4.2. *Let $M(x)$ and $M'(x')$ be as in Theorem* 4.1. *Fix any numbers $l,\tau,k\geqslant 0$ with $\tau+l\leqslant k$ and a jet $\Lambda_0\in D_{M(0),0;M'(0),0}^{\tau+l,l}$. Then there exist a neighborhood*

$$\tilde\Omega = \tilde\Omega(0, 0, \Lambda_0; 0, 0)\subset\mathbb{C}^N\times\mathbb{C}^N\times J_{0,0}^{\tau+l}(\mathbb{C}^N,\mathbb{C}^{N'})\times U\times U'$$

and a $\mathscr{C}^\kappa$ map

$$\Psi^{k,\tau}(\zeta, Z, \Lambda; x, x'): \tilde\Omega\to J^\tau(\mathbb{C}^N,\mathbb{C}^{N'}),$$

holomorphic in $(\zeta, Z, \Lambda)\in\mathbb{C}^N\times J_{0,0}^r(\mathbb{C}^N,\mathbb{C}^{N'})$ such that the following holds:

(i) *For the image $\Lambda_0^\tau\in J_{0,0}^\tau(\mathbb{C}^N,\mathbb{C}^{N'})$ of Λ_0 under the projection $J^{\tau+l}(\mathbb{C}^N,\mathbb{C}^{N'})\to J^\tau(\mathbb{C}^N,\mathbb{C}^{N'})$, one has $\overline{\Lambda_0^\tau} = \Psi^{k,\tau}(0, 0, \Lambda_0; 0, 0)$.*

(ii) *For every $x\in U$ and $x'\in U'$, every holomorphic map F from a neighborhood of* 0 *in $\mathbb{C}^N$ into $\mathbb{C}^{N'}$ with $(0, 0, j_0^r F; x, x')\in\tilde\Omega$ that sends* 0 *into* 0 *and $M(x)$ into $M'(x')$ up to order k at* 0, *satisfies the identity*

$$j_\zeta^\tau\overline{F} = \Psi^{k,\tau}(\zeta, Z, j_Z^{\tau+l} F; x, x') + o(|(Z,\zeta)|^{k-l-\tau}) \quad \textit{as } (Z,\zeta)\to 0 \textit{ in } \mathscr{M}(x). \tag{4.10}$$

If $M(x)$ and $M'(x')$ are in addition real-analytic, the maps $\Psi^{k,\tau}$ can be also chosen to be real-analytic.

4.2. Singular jet parametrization

Our next step will be to iterate the identity (4.10) as in [4,11,61]. The property (i) in Proposition 4.2 allows us to iterate it for $(Z, j_Z^r F; x, x')$ sufficiently close to $(0, \Lambda_0; 0, 0)$. After iterating (4.10) $2(d+1)$ times we obtain

$$F(\zeta^0) = \Phi^k(\zeta^0, \dots, \zeta^{2(d+1)}, j^r_{\zeta^{2(d+1)}} F; x, x') + o(|(\zeta^0, \zeta^1, \dots, \zeta^{2(d+1)})|^{k-r}) \tag{4.11}$$

as $(\zeta^0, \dots, \zeta^{2(d+1)}) \to 0$ in $\mathscr{M}^{2(d+1)}(x)$, where $r = 2(d+1)l$ as before and $\Phi^k(\zeta^0, \dots, \zeta^{2(d+1)}, \Lambda; x, x')$ is of class $\mathscr{C}^\kappa$ and holomorphic in $(\zeta^0, \dots, \zeta^{2(d+1)}, \Lambda)$.

Now let δ and φ be as in Lemma 3.4. Put $\zeta^{2(d+1)} = 0$ and

$$\zeta^0 = Z = V\left(\eta, \varphi\left(\eta, \frac{Z}{\delta(\eta; x)}; x\right); x\right)$$

in the identity (4.11). We obtain

$$F(Z) = \tilde{\Psi}^k\left(\eta, \frac{Z}{\delta(\eta; x)}, j_0^r F; x, x'\right) + R_F^k\left(\eta, \frac{Z}{\delta(\eta; x)}; x, x'\right), \tag{4.12}$$

where $\tilde{\Psi}^k$ and R_F^k are of class $\mathscr{C}^\kappa$, their restrictions $\tilde{\Psi}^k(\cdot, \cdot, \cdot; x, x')$ and $R_F^k(\cdot; x, x')$ are holomorphic and

$$R_F^k(\eta, \tilde{Z}; x, x') = o(|(\eta, \tilde{Z})|^{k-r}) \quad \text{as } (\eta, \tilde{Z}) \to 0 \tag{4.13}$$

for each fixed (x, x'). We note that here, exactly as in [11], only R_F^s (but not $\tilde{\Psi}^s$) depends on F (this dependence is indicated by F in the subscript). A new ingredient comparing with [11] is that $\tilde{\Psi}^k$ is merely smooth and depends on the choice of the families $M(x)$ and $M'(x')$.

Let $\eta_0 \in \mathbb{C}^{sn}$ be given by Lemma 3.5 and define $\hat{\delta}(\lambda; x) := \delta(\lambda\eta_0; x)$ for $\lambda \in \mathbb{C}$ near the origin. Then, setting $\eta = \lambda\eta_0$ in (4.12), we have

$$F(Z) = \hat{\Psi}^k\left(\lambda, \frac{Z}{\hat{\delta}(\lambda; x)}, j_0^r F; x, x'\right) + \hat{R}_F^k\left(\lambda, \frac{Z}{\hat{\delta}(\lambda; x)}; x, x'\right), \tag{4.14}$$

where $\hat{\Psi}^k$ and $\hat{R}_F^k$ are defined in the obvious way and have the properties analogous to those of $\tilde{\Psi}^k$ and R_F^k respectively. After a holomorphic change of the parameter $\lambda \in \mathbb{C}$, we may assume that $\hat{\delta}(\lambda; 0) = \lambda^m$, where $m \leqslant 2d(\nu - 1)$ is the vanishing order considered in Lemma 3.5. We shall view the vector $x \in \mathbb{R}^s$ as a row. Then

$$\hat{\delta}(\lambda; x) \equiv \lambda^m + a(\lambda; x)x, \tag{4.15}$$

where $a(\lambda; x)$ is a $\mathscr{C}^{\kappa-1}$ vector column which is holomorphic for each fixed x. Then (4.14) implies

$$F(Z) = \hat{\Psi}^k\left(\lambda, \frac{Z}{\lambda^m}\left(1 + \frac{a(\lambda; x)x}{\lambda^m}\right)^{-1}, j_0^r F; x, x'\right) + \hat{R}_F^k\left(\lambda, \frac{Z}{\hat{\delta}(\lambda; x)}; x, x'\right). \tag{4.16}$$

Note that the decomposition (4.15) is used only for the first term on the right-hand side. Such "trick" will be essential for proving smoothness of the first term after resolving the singularity. As for the second

term, we shall need only an estimate on the vanishing order for each fixed (x, x') that will be obtained directly. We now rewrite (4.16) in the form

$$F(Z) = \check{\Psi}^k\left(\lambda, \frac{Z}{\lambda^m}, \frac{x}{\lambda^m}, j_0^r F; x, x'\right) + \hat{R}_F^k\left(\lambda, \frac{Z}{\hat{\delta}(\lambda; x)}; x, x'\right), \tag{4.17}$$

where $\check{\Psi}^k(\lambda, \tilde{Z}, \tilde{x}, \Lambda; x, x')$ is defined and $\mathscr{C}^{\kappa-1}$ for $(\lambda, \tilde{Z}, \tilde{x}, \Lambda; x, x')$ in a neighborhood of $(0, 0, 0, \Lambda_0; 0, 0)$ and holomorphic in $(\lambda, \tilde{Z}, \tilde{x}, \Lambda)$. This is the *singular jet parametrization* that we shall use for the proof of Theorem 4.1.

4.3. Resolution of singularities with smooth parameters

We will apply the following statement to the map $\check{\Psi}^k$ above that can be seen as a variant of Lemma 10.6 in [11] with smooth parameters.

Lemma 4.3. *Let $P(\lambda, \tilde{t}; x)$ be a function of class $\mathscr{C}^\tau$ in a neighborhood of* 0 *in $\mathbb{C} \times \mathbb{C}^n \times \mathbb{R}^h$ such that $P(\cdot, \cdot; x)$ is holomorphic for each fixed x. Let*

$$P\left(\lambda, \frac{t}{\lambda^m}; x\right) = \sum_\nu c_\nu(t; x)\lambda^\nu \tag{4.18}$$

be the Laurent series expansion in λ. Then the coefficient c_0 is of class $\mathscr{C}^\tau$ in a neighborhood of the origin and is holomorphic there in t for each fixed x. If furthermore P is real-analytic in all variables, so is c_0.

Proof. Since $P(\cdot, \cdot; x)$ is holomorphic, we can write

$$P(\lambda, \tilde{t}; x) = \sum_{\nu,\alpha} P_{\nu,\alpha}(x)\lambda^\nu \tilde{t}^\alpha$$

in a neighborhood of the origin with $|P_{\nu,\alpha}(x)| < C^{\nu+|\alpha|}$ for some constant $C > 0$. Then $P_{\nu,\alpha} \in \mathscr{C}^\tau$ and

$$c_0(t; x) = \sum_{\nu=|\alpha|m} P_{\nu,\alpha}(x)t^\alpha \tag{4.19}$$

is clearly holomorphic in t for $(t; x)$ near the origin. Since partial differentiations commute, all partial derivatives of $P(\lambda, \tilde{t}; x)$ with respect to x are holomorphic in the variables $(\lambda, \tilde{t})$:

$$\bar{\partial}_{(\lambda,\tilde{t})}\partial_x^\beta P = \partial_x^\beta \bar{\partial}_{(\lambda,\tilde{t})} P = 0$$

for every $|\beta| \leqslant \tau$. Furthermore the power series

$$\partial_x^\beta P(\lambda, \tilde{t}; x) = \sum_{\nu,\alpha} \partial_x^\beta P_{\nu,\alpha}(x)\lambda^\nu \tilde{t}^\alpha$$

converge absolutely for all $|\beta| \leqslant \tau$ and for $(\lambda, \tilde{t})$ near the origin. Then

$$\partial_x^\beta c_0(t; x) = \sum_{\nu=|\alpha|m} \partial_x^\beta P_{\nu,\alpha}(x)t^\alpha,$$

shows that $c_0(t;x)$ is of class $\mathscr{C}^\tau$ as required. Finally, if P is real-analytic, it can be further expanded near 0 as a convergent power series in (t,x). Then c_0 must be real-analytic as a consequence of (4.19). □

The following lemma will be used for the function $\hat{R}_F^k$ in the right-hand side of (4.17) and can be also seen as a variant of Lemma 10.6 in [11] with smooth parameters.

Lemma 4.4. *Let $R(\lambda,\tilde{t})$ and $\delta(\lambda)$ be holomorphic functions near the origins in $\mathbb{C}\times\mathbb{C}^n$ and in $\mathbb{C}$ respectively such that*

$$R(\lambda,\tilde{t}) = o(|(\lambda,\tilde{t})|^h) \quad as\ (\lambda,\tilde{t})\to 0$$

and suppose that $\delta(\lambda)$ has the vanishing order g at $\lambda=0$ for some nonnegative integers h and g. Consider the Laurent series expansion

$$R\left(\lambda,\frac{t}{\delta(\lambda)}\right) = \sum_\nu c_\nu(t)\lambda^\nu,$$

where each $c_\nu(t)$ is a formal power series obtained by expanding the left-hand side. Then $c_0(t)=o(|t|^{\frac{h}{g+1}})$ as $t\to 0$.

Proof. We write R and $t/\delta(\lambda)$ as formal power series:

$$R(\lambda,\tilde{t}) = \sum_{\nu,\alpha} R_{\nu,\alpha}\lambda^\nu\tilde{t}^\alpha, \qquad \frac{t}{\delta(\lambda)} = t\lambda^{-g}(a_0+a_1\lambda+a_2\lambda^2+\cdots),$$

where $a_0\neq 0$. The substitution $\tilde{t}=t/\delta(\lambda)$ yields

$$R\left(\lambda,\frac{t}{\delta(\lambda)}\right) = \sum_{\nu,\alpha} R_{\nu,\alpha}t^\alpha\lambda^{\nu-g|\alpha|}(a_0+a_1\lambda+a_2\lambda^2+\cdots)^\alpha.$$

Then it is easy to see that $c_0(t)=\sum b_\alpha t^\alpha$ is a formal power series such that each coefficient b_α is a finite linear combinations of $R_{\nu,\alpha}$ for $\nu-g|\alpha|\leqslant 0$. By the assumption, $R(\lambda,\tilde{t})=o(h)$ and hence the inequality $\nu+|\alpha|>h$ holds for all nonvanishing coefficients $R_{\nu,\alpha}$. Putting these inequalities together we obtain $|\alpha|>h/(g+1)$ showing $c_0(t)=o(h/(g+1))$ as required. □

4.4. The end of proof of Theorem 4.1

Let $M(x), M'(x'), \kappa, \nu, d, l, r$ and Λ_0 be as in Theorem 4.1. For each $0\leqslant\tau\leqslant r$, denote by $\Lambda_0^\tau\in J^\tau(\mathbb{C}^N,\mathbb{C}^{N'})$ as before the image of Λ_0 under the projection $J^r(\mathbb{C}^N,\mathbb{C}^{N'})\to J^\tau(\mathbb{C}^N,\mathbb{C}^{N'})$. Consider the the Laurent series expansion in λ of the function $\check{\Psi}^k$ in (4.17), where we substitute Z/λ^m for the second argument as in (4.17) and t/λ^m for the third argument with $t\in\mathbb{R}^s$:

$$\check{\Psi}^k\left(\lambda,\frac{Z}{\lambda^m},\frac{t}{\lambda^m},\Lambda;x,x'\right) = \sum_\nu c_\nu^k(Z,t,\Lambda;x,x')\lambda^\nu. \tag{4.20}$$

By Lemma 4.3, $c_0^k(Z, t, \Lambda; x, x')$ is of class $\mathscr{C}^\kappa$ and holomorphic in (Z, t, Λ) for each fixed (x, x'). We obtain the $\mathscr{C}^\kappa$ map

$$\Psi^k(Z, \Lambda; x, x') := c_0^k(Z, x, \Lambda; x, x')$$

defined in a neighborhood of $(0, \Lambda_0; 0, 0)$ and holomorphic in (Z, Λ). It remains to show (4.2). For this, consider the Laurent series expansions of both sides of (4.17) in λ. Since

$$\hat{R}_F^k(\lambda, \tilde{Z}; x, x') = o(|(\lambda, \tilde{Z})|^{k-r}) \quad \text{as } (\lambda, \tilde{Z}) \to 0$$

by (4.13) and the construction of $\hat{R}_F^k$, it follows from Lemma 4.4 that the constant term c_0 in λ in the Laurent series expansion of

$$\hat{R}_F^k\left(\lambda, \frac{Z}{\hat{\delta}(\lambda; x)}; x, x'\right) \tag{4.21}$$

is $o(k - r/m_x + 1)$ in Z for fixed (x, x'), where m_x is the vanishing order of $\hat{\delta}(\lambda; x)$ at $\lambda = 0$. Clearly we have $m_x \leqslant m$ for x close to 0, where $m = m_0$ is the vanishing order of $\hat{\delta}(\lambda; 0)$. Therefore the expression in (4.21) is always at least $o(k - r/m + 1)$. By equating the constant terms (in λ) in the Laurent series expansion of (4.17) we obtain the required identity (4.2). The proof of Theorem 4.1 is complete.

5. Applications of Theorem 4.1: proofs of the main results

Proof of Theorem 2.1. We keep here all notations and assumptions from Theorem 2.1. We first construct a section Φ locally near a fixed jet $\lambda_0 \in D^{r,l}(M, M')$ with a source $x_0 \in M$ and a target $x_0' \in M'$. For this, we apply Proposition 3.1 to both M and M' to obtain, for the given integer k, $\mathscr{C}^{\kappa-k}$ families $M(x) = \varphi_x(U)$, $M'(x') = \varphi'_{x'}(U')$, $(x, x') \in U \times U' \subset M \times M'$, of local real-analytic approximate embeddings of order k of M and M' into $\mathbb{C}^N$ and $\mathbb{C}^{N'}$ respectively passing through 0. Fixing real coordinates on M and M' vanishing at x_0 and x_0' we may identify U and U' (after possible shrinking) with open neighborhoods of 0 in $\mathbb{R}^s$ and $\mathbb{R}^{s'}$ respectively with $s := \dim_{\mathbb{R}} M$ and $s' := \dim_{\mathbb{R}} M'$.

Next we use the diffeomorphisms $\varphi_0 \colon U \to M(0)$ and $\varphi_0' \colon U' \to M'(0)$ to transfer $\lambda_0 \in J_{0,0}^r(M, M')$ to a jet $\tilde{\lambda}_0 \in J_{0,0}^r(M(x_0), M'(x_0'))$ by setting $\tilde{\lambda}_0 := j_0^r(\varphi_0' \circ f \circ \varphi_0^{-1})$, where f is any representative of λ_0. Since both φ_0 and φ_0' are CR up to order k at the origins, it follows that $\tilde{\lambda}_0 \in D^{r,l}(M(0), M'(0))$. Similarly we consider, for each (x, x'), the one-to-one transformation sending a jet $\lambda \in J_{(x,x')}^r(M, M')$ represented by a map f to the jet

$$\tilde{\lambda} := j_0^r(\varphi'_{x'} \circ f \circ \varphi_x^{-1}) \in J_{0,0}^r(M(x), M'(x')).$$

Furthermore, as indicated at the beginning of Section 4, any CR r-jet $\tilde{\lambda} \in J_{0,0}^r(M(x), M'(x'))$ extends uniquely to a holomorphic r-jet $\Lambda \in J_{0,0}^r(\mathbb{C}^N, \mathbb{C}^{N'})$ sending $(M(x), 0)$ into $(M'(x'), 0)$ (the uniqueness follows from the property that $M(x)$ is generic in $\mathbb{C}^N$). We can now write $\Lambda = \Xi_1(\lambda; x, x')$, where, in local coordinates, the components of $\Xi_1(\lambda; x, x')$ can be written as polynomials in the components of λ whose coefficients are $\mathscr{C}^{\kappa-r}$ in (x, x'). Hence Ξ_1 is a $\mathscr{C}^{\kappa-r}$ map from a neighborhood of $(\lambda_0, 0, 0)$ in $J^r(M, M')$ into $J_{0,0}^r(\mathbb{C}^N, \mathbb{C}^{N'})$. By the construction, the jet $\Lambda_0 := \Xi_1(\lambda_0; x, x')$ is in $D^{r,l}_{M(0),0;M'(0),0}$,

where as in Section 4, $D^{r,l}_{M(x_0),0;M'(x'_0),0}$ denotes the set of all r-jets sending $(M(0),0)$ into $(M'(0),0)$ that are $\tilde{l}$-nondegenerate for some $\tilde{l} \leqslant l$.

We can now apply Theorem 4.1 (with κ replaced by $\kappa - k$) to the families $M(x)$ and $M'(x')$ and the jet Λ_0. Let $\Psi^k : \Omega \to \mathbb{C}^{N'}$ be the $\mathscr{C}^{\kappa-k-1}$ map given by this theorem, where Ω is an open neighborhood of $(0, \Lambda_0, 0, 0)$ in $\mathbb{C}^N \times J^r(\mathbb{C}^N, \mathbb{C}^{N'}) \times U \times U'$. Our choice of r and k in (2.3) and the inequality $m \leqslant 2d(\nu - 1)$ in Lemma 3.5 imply that $(k-r)/(m+1) \geqslant r+1$. Hence Theorem 4.1 gives a parametrization of jets of order $r+1$ of holomorphic maps from $\mathbb{C}^N$ to $\mathbb{C}^{N'}$ that we have to "transform" back to jets of the same order of smooth maps between M and M'. In other words we differentiate the identity (4.2) $r+1$ times in Z and evaluate at $Z = x$. We obtain an identity $j_0^{r+1} F = \Psi(j_0^r F; x, x')$, where Ψ can be chosen to be a family of local sections of the bundle $J^{r+1}_{(0,0)}(\mathbb{C}^N, \mathbb{C}^{N'}) \to J^r_{(0,0)}(\mathbb{C}^N, \mathbb{C}^{N'})$ defined in a neighborhood of Λ_0. Moreover, since Ψ is obtained by differentiating Ψ^k only in Z, where Ψ^k is holomorphic, it follows from the Cauchy formula that Ψ has the same smoothness as Ψ^k.

If now f satisfies the assumptions of Theorem 2.1 and $F : \mathbb{C}^N \to \mathbb{C}^{N'}$ a holomorphic extension of $\varphi'_{x'} \circ f \circ \varphi_x^{-1}$ as above, it follows again from the chain rule that $j_x^{r+1} f = \Xi_2(j_0^{r+1} F; x, x')$, where Ξ_2 is a $\mathscr{C}^{\kappa-k-r-1}$ function in a neighborhood of $(\Lambda_0; 0, 0)$. By composing Ψ with Ξ_1 and Ξ_2 we obtain $j_x^{r+1} f = \tilde{\Phi}(j_x^r f)$, where $\tilde{\Phi}$ is a $\mathscr{C}^{\kappa-k-r-1}$ $J^{r+1}(M, M')$-valued function defined in a neighborhood of (x_0, λ_0). Clearly $\tilde{\Phi}$ can be assumed to be a local section of the fiber bundle $\pi : J^{r+1}(M, M') \to J^r(M, M')$.

In order to obtain a global section Φ of π over $D^{r,l}(M, M')$ as claimed we use the fact that π carries a canonical affine bundle structure. More precisely, one can use standard trivializations of π induced by local coordinates on M and M'. Then it follows from the chain rule that the transition maps are affine on the fibers of π. Hence linear combinations of sections of π are well-defined provided the sum of the coefficients is 1. Hence we can use a partition of unity in $D^{r,l}(M, M')$ to glue the above local sections to obtain a global section Φ satisfying the required properties. □

Proof of Corollary 2.2. Since the main argument seems to be well-known, we only give a sketch of the proof (see e.g. [10,29] for a similar argument). Let Φ be given by Theorem 2.1 and choose local coordinates $x \in \mathbb{R}^{\dim M}$ on M and $x' \in \mathbb{R}^{\dim M'}$ on M' vanishing respectively at the source $x_0 \in M$ and the target x'_0 of the jet λ_0. Then the differential system given by (2.4) implies a system of differential equations for $\Lambda(x) := j_x^r f$ of the form

$$\partial_{x_j} \Lambda(x) = \Phi_j(x, \Lambda(x)). \tag{5.1}$$

We obtain the required map Θ by integrating (5.1) along paths. More precisely, every points $p, x \in M$ sufficiently close to x_0 can be connected in a unique way by a sequence of paths $\gamma_1, \ldots, \gamma_{\dim M}$ such that each γ_j is an integral curve of $\partial/\partial x_j$. Then we define $\Theta(x, p, \lambda)$ to be the result of solving the ODE obtained from (5.1) along this uniquely defined chain of paths. It is easy to see that Θ satisfies the required conclusion. □

Proof of Theorem 1.4. The statement is a consequence of the local parametrization provided by Corollary 1.3. Indeed, given the map $\Psi(x, \Lambda)$ as in this corollary, put $\Psi_\Lambda := \Psi(\cdot, \Lambda)$. Then, if f and q are as in Corollary 1.3 and $\Lambda := j_q^r f$, then $\Psi_\Lambda \equiv f$ is CR in a neighborhood of q. On the other hand, if Ψ_Λ is CR and

$$j_q^r \Psi_\Lambda = \Lambda, \tag{5.2}$$

we obtain a local CR-diffeomorphism $f := \Psi_\Lambda$ between M and M' defined in a neighborhood of q and having its k-jet there equal to Λ. Hence the system of tangential Cauchy-Riemann equations for Ψ_Λ together with (5.2) give necessary and sufficient conditions for the existence of a CR-diffeomorphism with prescribed jet as in the statement of the theorem. Clearly both conditions can be expressed in terms of vanishing of a collection of real functions $\varphi_1(x, \Lambda), \ldots, \varphi_s(x, \Lambda)$. To finish the proof, it remains to set $\Gamma := \varphi_1^2 + \cdots + \varphi_s^2$. $\square$

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