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Non-embeddable CR-manifolds of higher codimension

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Abstract. For all integers $d \geqq k \geqq 1$ and n suitably large we give explicit examples of connected compact real-analytic submanifolds $M \subset \mathbb{C}^n$ with the following properties: (1) Every non-trivial covering space of M is non-embeddable in the sense that it is not CR-isomorphic (with respect to its canonical CR-structure) to a CR-submanifold of $\mathbb{C}^N$ for any N whatsoever. (2) M has fundamental group $\pi_1(M) \cong \mathbb{Z}_2^k$, where $\mathbb{Z}_2$ is the group of order two. (3) The covering spaces of M, indexed by the subgroups of $\pi_1(M)$, are pairwise CR-non-isomorphic. (4) M is a strongly pseudoconvex Cauchy-Riemann submanifold with CR-codimension $\geqq d$. (5) M is homogeneous with respect to a compact linear subgroup $G \subset \mathsf{GL}(n, \mathbb{C})$. (6) M is not locally a direct product of CR-manifolds of lower dimensions.

1. Introduction

It is well-known that every real-analytic CR-manifold is locally CR-embeddable into $\mathbb{C}^n$ for some n, see for instance [2]. In contrast to this, the question about *global* embeddability into a suitable $\mathbb{C}^n$ has a negative answer in general. One can obtain fairly general classes of globally non-embeddable CR-manifolds by requiring a certain *pseudoconcavity* property. This roughly means that at every point the Levi form spans all directions. Further examples occur more generally among CR-manifolds satisfying the so-called *strong maximum principle* for continuous CR-functions, see [7], [8], [14], [12]. On every compact connected CR-manifold of this type all CR-functions are constant, thus implying non-embeddability. Here and throughout the paper we call a CR-manifold M *non-embeddable* if it is not isomorphic to a CR-submanifold of any $\mathbb{C}^n$—note that every real-analytic CR-manifold is always embeddable into some complex manifold [1]. Because of the above we are only interested in non-embeddable CR-manifolds with 'many' non-constant CR-functions.

In case the CR-manifold M is strongly pseudoconvex, the maximum principle does not hold in general. Here *strongly pseudoconvex* means in case M is of hypersurface type that the Levi form is definite at every point of M and, in the general case, that the Levi form

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at every point is positively definite with respect to some conormal (see e.g. [18], [10], [17]). The first non-embeddable example of this type (attributed to Andreotti in [16], see also [3]) is compact and has dimension 3. Later, other interesting 3-dimensional non-embeddable examples have been discovered (see e.g. [9]), whereas in dimension > 3 compact strongly pseudoconvex CR-manifolds of hypersurface type are necessarily embeddable due to [4] (see also [16] for the real-analytic case). To the authors' knowledge, all known non-embeddable strongly pseudoconvex CR-manifolds so far have been essentially of hypersurface type. By "essentially" here we mean to exclude the trivial way of producing further examples by taking direct products of two CR-manifolds, one of which is non-embeddable.

In this paper we construct multi-parameter series of explicit examples of compact non-embeddable real-analytic strongly pseudoconvex CR-manifolds of *arbitrary* CR-*codimension* that are not locally products of lower-dimensional CR-manifolds. All these examples are homogeneous with respect to a compact Lie group and have many non-constant CR-functions in the sense that they are finite covers of embeddable CR-manifolds.

2. Description of the examples

Let G be a Lie group. A *linear G-space* is a complex linear space E of finite dimension together with a continuous representation $\Phi : G \to \mathsf{GL}(E)$—instead of $\Phi(g)(a)$ for $g \in G$ and $a \in E$ we simply write $g \cdot a$. In case G is compact, every orbit $G \cdot a$, $a \in E$, is a real-analytic CR-submanifold of E, on which G acts transitively and analytically by CR-transformations.

For our examples we fix an integer $p \geqq 2$ and let E be the linear space of all symmetric complex $p \times p$ matrices, that is, $E = \{z \in \mathbb{C}^{p\times p} : z' = z\}$, where z' denotes the transpose of the matrix z. Then E is a linear G-space for $G := \mathsf{SU}(p)$ if we put $g \cdot z := gzg'$ for all $g \in G$ and $z \in E$. For every $a \in E$ the corresponding orbit has the following explicit description:

$$G \cdot a = \{z \in E : \det(z) = \det(a) \text{ and } m_j(z) = m_j(a) \text{ for all } j < p\}, \tag{2.1}$$

where $m_j(z)$ is the sum over all $j \times j$-diagonal minors of the matrix $zz^* \in \mathbb{C}^{p\times p}$, see [15], also for the following. Consider the $(p-1)$-dimensional simplex

$$\Delta := \{x \in \mathbb{R}^p : 1 = x_1 \geqq \cdots \geqq x_p \geqq 0\} \tag{2.2}$$

and identify every $a \in \Delta$ in the canonical way with the corresponding diagonal matrix in E (having diagonal entries $a_{ii} = a_i$ for $1 \leqq i \leqq p$). Then for every $0 \neq e \in E$ the orbit $G \cdot e$ is CR-isomorphic to some orbit $G \cdot a$, $a \in \Delta$, via a suitable homothety $z \mapsto \alpha z$, $\alpha \in \mathbb{C}^*$. Let

$$\Delta^+ := \{x \in \Delta : x_p > 0\} \quad \text{and} \quad \Delta^0 := \{x \in \Delta : x_p = 0\} \tag{2.3}$$

be the subsets of invertible and non-invertible matrices in Δ respectively. Then

$$(x_1, x_2, \ldots, x_p) \mapsto \left(\frac{x_p}{x_p}, \frac{x_p}{x_{p-1}}, \ldots, \frac{x_p}{x_1}\right) \tag{2.4}$$

defines a homeomorphism $\theta : \Delta^+ \to \Delta^+$ of period 2. In case $p = 2$ the transformation θ is the identity on Δ^+, in all other cases $\mathsf{Fix}(\theta)$ is a proper (but non-empty) subset of Δ^+.

The CR-equivalence problem for G-orbits in E (actually for a much bigger class of examples) has been solved in [15], section 13: *The orbits* $G \cdot a$ *and* $G \cdot b$ *for* $a \neq b$ *from* Δ *are* CR-*isomorphic if and only if* $a \in \Delta^+$ *and* $b = \theta(a)$. Notice that for every $a \in \Delta^+$, $b := \theta(a)$ and $\alpha := a_p > 0$

$$\theta_a : G \cdot a \to G \cdot b, \quad z \mapsto \alpha z^{-1}, \tag{2.5}$$

defines a CR-diffeomorphism between the two orbits.

For the unit matrix $\mathbb{1} = (1, \dots, 1) \in \Delta$, the orbit $G \cdot \mathbb{1} = \mathsf{SU}(p) \cap E$ is totally real in E. For every other $a \in \Delta$, the orbit $G \cdot a$ is a minimal, strongly pseudoconvex CR-manifold, and from the explicit description of the Levi form at a, compare [15], section 9, it can be seen that $G \cdot a$ locally is not the direct product of CR-manifolds of lower dimensions.

For every $a \in \Delta$ let $k = k(a)$ be the maximal number of pairwise different coordinates of a. Then the orbit $M := G \cdot a$ has CR-codimension $\geqq (k - 1)$ and fundamental group $\mathbb{Z}_2^{k-1}$, where $\mathbb{Z}_2$ is the group of order 2, see Sect. 4. The universal covering $\tilde{M}$ of M is a CR-manifold in a natural way. We show that $\tilde{M}$ is not separable by continuous CR-functions in general, thus giving an example of a non-embeddable strongly pseudoconvex CR-manifold. To be more specific, we show, for instance, for all $a \in \Delta^+$ with $\theta(a) \neq a$: *Every non-trivial covering of the orbit* $M := G \cdot a$ *is not separable by continuous* CR-*functions. Furthermore, the covering spaces of* M *are pairwise non-isomorphic as* CR-*manifolds.* Recall that the coverings of M are in 1:1-correspondence with the subgroups of the homotopy group $\pi_1(M, a) \simeq \mathbb{Z}_2^{k-1}$ and that the number N_n of subgroups in $\mathbb{Z}_2^n$ satisfies the recursion formula: $N_0 = 1$, $N_1 = 2$ and $N_{n+1} = 2N_n + (2^n - 1)N_{n-1}$.

Since $G = \mathsf{SU}(p)$ is connected and simply-connected, the G-action on every $M = G \cdot a$ lifts to a G-action on the universal covering $\tilde{M}$ of M. In case $k(a) = p$, the universal covering $\tilde{M}$ is isomorphic as homogeneous G-space to the group G acting on itself by left translations. In particular, we can construct from this a $(p - 1)$-parameter family of pairwise CR-inequivalent strongly pseudoconvex leftinvariant CR-structures on $\mathsf{SU}(p)$ (see [15], section 13), each of which is non-embeddable.

We would like to mention that the classical tools used to show non-embeddability (see [16], [9]) are not available in higher codimension. In particular, a submanifold of higher codimension does not bound any domain and hence the question of finding a suitable domain of extension for CR-functions is more delicate. Here we use our results in [15] describing such regions of holomorphic extension. Furthermore, strong pseudoconvexity of the initial CR-manifold cannot be used to conclude that the ramification locus in such a region is empty. Our arguments here are based on the Peter-Weyl Theorem.

3. Some consequences of the Peter-Weyl Theorem

Let G be a compact Lie group and let M be an analytic CR-manifold on which G acts transitively and analytically by CR-transformations.

3.1. Proposition. *The following conditions are equivalent*:

(i) *M is embeddable.*

(ii) *The continuous* CR-*functions separate points on M.*

(iii) *There exists a linear G-space V together with a G-equivariant* CR-*isomorphism from M onto some orbit $G \cdot v$ in V.*

Proof. (i) $\Rightarrow$ (ii) and (iii) $\Rightarrow$ (i) are trivial. Suppose that (ii) holds and fix a point $a \in M$. Denote by $\mathscr{C}_{\mathrm{CR}}(M)$ the complex Banach algebra of all continuous CR-functions on M. Then G acts by linear isometries on $\mathscr{C}_{\mathrm{CR}}(M)$ if we associate to every $g \in G$ the linear operator $f \mapsto f \circ g^{-1}$. Denote by $\mathscr{R} \subset \mathscr{C}_{\mathrm{CR}}(M)$ the linear subspace of all representative functions on G, that is, of all $f \in \mathscr{C}_{\mathrm{CR}}(M)$ that are contained in some G-invariant linear subspace of finite dimension in $\mathscr{C}_{\mathrm{CR}}(M)$. By the Peter-Weyl Theorem, compare for instance [5], p. 141, $\mathscr{R}$ is a dense subalgebra of $\mathscr{C}_{\mathrm{CR}}(M)$. Define inductively finite chains

$$\{0\} = V_0 \subset V_1 \subset \cdots \subset V_k \quad \text{and} \quad M = M_0 \supset M_1 \supset \cdots \supset M_k = \{a\}$$

in the following way, where every V_j is a G-invariant linear subspace of finite dimension in $\mathscr{R}$ and $M_j = \{z \in M : f(z) = f(a) \text{ for all } f \in V_j\}$: Assume that V_j with $M_j \neq \{a\}$ is already defined. Let X be the subspace of all functions in $\mathscr{R}$ that are constant on M_j. By (i), X is not dense in $\mathscr{C}_{\mathrm{CR}}(M)$ and hence also is not dense in $\mathscr{R}$. Therefore there exists a function $f \in \mathscr{R} \backslash X$. Let V_{j+1} be the smallest G-invariant linear subspace of $\mathscr{R}$ that contains V_j and f. Then V_{j+1} has finite dimension and $M_{j+1} \neq M_j$. Since representative functions are known to be real-analytic and every properly descending chain of closed real-analytic subsets in M is finite, the induction stops after a finite number of steps k. The dual $V := \mathscr{L}(V_k, \mathbb{C})$ of V_k is a linear G-space with respect to $(g \cdot \lambda)(f) := \lambda(f \circ g)$ for all $g \in G$, $f \in W$, and $\varepsilon(z)(f) := f(z)$ defines a real-analytic CR-map $\varepsilon : M \to V$. Since ε is also G-equivariant we get a CR-isomorphism from M onto the orbit $G \cdot v$ for $v := \varepsilon(a)$, proving (iii). $\square$

In the following let M be a connected CR-manifold and let $\mathscr{C}_{\mathrm{CR}}(M)$ be the algebra of all continuous CR-functions on M. We always assume that $\mathscr{C}_{\mathrm{CR}}(M)$ separates the points of M and that G is a compact connected and simply-connected Lie group acting transitively and analytically by CR-transformations on M. This implies that G is semi-simple and that M has finite fundamental group. In case $\tau : N \to M$ is a covering map with a Hausdorff topological space N there is a unique structure of CR-manifold on N such that τ is a local CR-isomorphism and we ask: *When do the continuous* CR-*functions on N separate points*? or, what is equivalent in view of Proposition 3.1: *When is N embeddable*? For this consider on N the equivalence relation given by identifying points that cannot be separated by continuous CR-functions and denote by $\underline{N}$ the corresponding quotient space. Since M is separable by CR-functions, the covering map $\tau : N \to M$ factors over the canonical projection $N \to \underline{N}$ and a mapping $\underline{\tau} : \underline{N} \to M$. The action of G on M lifts to an action on N. From this it is easily derived that $\underline{N}$ is a (Hausdorff) CR-manifold, separable by CR-functions, and that both mappings are covering maps themselves. In case N is the universal covering of M we write $\tilde{M}$ instead of N and $\hat{M}$ instead of $\underline{N}$. It is clear that every non-trivial covering of $\hat{M}$ gives an example of a non-embeddable CR-manifold.

3.2. Proposition. *Let V be a linear G-space for a compact connected simply-connected Lie group G, let M be a G-orbit in V and let $\tau : N \to M$ be a covering map with N a connected Hausdorff space. Suppose there exists a locally-closed complex submanifold Y of V with the following properties*:

(i) *Every continuous* CR-*function on* M *has a unique continuous extension to* $M \cup Y$ *which is holomorphic on* Y.

(ii) *For every* $y \in Y$, *the orbit* $G \cdot y$ *is a Zariski-dense subset of* Y (*i.e.* $A = Y$ *is the only complex-analytic subset of* Y *with* $G \cdot y \subset A$).

(iii) $M \cup Y$ *is simply-connected.*

Then every continuous CR-*function on* N *is the pullback of a function from* M, *that is,* $\underline{N} = M$.

Proposition 3.2 is a special case of the following more general principle.

3.3. Proposition. *Let* $G, V, \tau : N \to M$ *and* Y *be as in Proposition* 3.2 *except that* (iii) *not necessarily holds. Suppose that* $a, b \in N$ *are points that can be connected by a continuous path* γ *in* N *whose projection to* M *is a null-homotopic loop in* $M \cup Y$. *Then* a, b *cannot be separated by continuous* CR-*functions on* N.

Proof. For $Y = \emptyset$ the statement obviously is true, so we assume $Y \neq \emptyset$ in the following. The action of G on M lifts in a unique way to a transitive action on the covering space $\underline{N}$ of M. By Proposition 3.1, $\underline{N}$ can be realized as a G-orbit in some linear G-space W. Denote by $d \geqq 1$ the degree of the covering map $\underline{\tau} : \underline{N} \to M$ (which is finite because the compact group G acts transitively on N) and let $X := W^d/\mathfrak{S}_d$ be the d^{th} symmetric power of W, where $\mathfrak{S}_d$ is the symmetric group in d elements. G acts in a canonical way by biholomorphic transformations on the complex space X and the canonical projection $\pi : W^d \to X$ is holomorphic. Denote by $B \subset X$ the ramification locus of this projection. Then $z \mapsto \underline{\tau}^{-1}(z)$ induces a continuous G-equivariant CR-map $\psi : M \to X \backslash B$. Since X can be realized as an analytic subset of some $\mathbb{C}^n$ we conclude with (i) that ψ extends to a continuous G-equivariant map $M \cup Y \to X$ whose restriction to Y is holomorphic. The preimage $\psi^{-1}(B)$ is a proper G-invariant analytic subset of Y and hence empty because of (ii). Therefore the covering $\underline{\tau} : \underline{N} \to M$ extends to a (non-ramified) covering $\eta : Z \to M \cup Y$ with $Z \subset W$ being a suitable subset with $\underline{N} \subset Z$, in such a way that $\psi(x) = \eta^{-1}(x)$ holds for all $x \in M \cup Y$. Since, by the assumptions, the projection of the path γ is a null-homotopic loop in $Y \cup M$, the projections of the two endpoints a and b to $\underline{N}$ must coincide. The required conclusion now follows directly from the construction of $\underline{N}$. □

In Proposition 3.3 the degree of the covering $\underline{\tau} : \underline{N} \to M$ is bounded by the order of the pointed fundamental group $\pi_1(M \cup Y, c)$, for every $c \in M$. Indeed, as a consequence of Proposition 3.3 for every $\underline{a} \in \underline{N}$ and $c := \underline{\tau}(\underline{a})$ the kernel of the canonical homomorphism $\pi_1(M, c) \to \pi_1(M \cup Y, c)$ contains the subgroup $\pi_1(\underline{N}, \underline{a})$ of $\pi_1(M, c)$.

4. Orbits of invertible matrices

We start by describing the examples of Sect. 2 in more detail: For fixed integer $p \geqq 2$ and $G := \mathsf{SU}(p)$, we again consider $E := \{z \in \mathbb{C}^{p \times p} : z = z'\}$ as linear G-space with respect to $g \cdot z := gzg'$ for all $g \in G$. For every $z \in E$ denote by

$$\sigma_1(z) \geqq \sigma_2(z) \geqq \cdots \geqq \sigma_p(z) \geqq 0$$

the eigenvalues of the non-negative hermitian matrix $\sqrt{zz^*}$, each counted according to its multiplicity. Every $\sigma_j(z)$ is called the j^{th} *singular value* of z. The function tuple $\sigma := (\sigma_1, \sigma_2, \dots, \sigma_p)$ defines a continuous G-invariant mapping $\sigma : E \to \mathbb{R}^p$ and

$$G \cdot a = \{z \in E : \det(z) = \det(a) \text{ and } \sigma(z) = \sigma(a)\}$$

for all $a \in E$, compare also (2.1).

Now fix an element $a \neq 0$ in E for the sequel and put $M := G \cdot a$. Denote, as before, by $\mathsf{k} := \mathsf{k}(a)$ the maximal number of pairwise different singular values of a, that is,

$$\{\sigma_1(a), \sigma_2(a), \dots, \sigma_p(a)\} = \{\lambda_1, \lambda_2, \dots, \lambda_{\mathsf{k}}\} \tag{4.1}$$

with $\lambda_1 > \lambda_2 > \cdots > \lambda_{\mathsf{k}} \geqq 0$. For every $j \leqq \mathsf{k}$, denote by $r_j \geqq 1$ the multiplicity of the singular value λ_j for a.

The connected and simply-connected Lie group $G = \mathsf{SU}(p)$ acts transitively on the CR-manifold M by CR-transformations. To apply the results from Sect. 3 we need suitable non-trivial coverings of M. We claim that the fundamental group of M is $\mathbb{Z}_2^{\mathsf{k}-1}$, where $\mathbb{Z}_2$ is the group with 2 elements. Indeed, if we assume $a \in \Delta$ without loss of generality, it can be shown that the isotropy subgroup $K := \{g \in G : g \cdot a = a\}$ is the subgroup

$$\mathsf{S}\big(\mathsf{O}(r_1) \times \mathsf{O}(r_2) \times \cdots \times \mathsf{O}(r_{\mathsf{k}})\big) \subset \mathsf{SU}(p), \tag{4.2}$$

where $\mathsf{S}(H) := \{g \in H : \det(h) = 1\}$ for every subgroup $H \subset \mathsf{GL}(p, \mathbb{C})$ and $\mathsf{O}(r) \subset \mathsf{GL}(r, \mathbb{R})$ for every integer $r \geqq 1$ is the orthogonal subgroup. The connected identity component K^0 of K is $\mathsf{SO}(r_1) \times \mathsf{SO}(r_2) \times \cdots \times \mathsf{SO}(r_{\mathsf{k}})$, that is, $\pi_1(M) \cong K/K^0$ is isomorphic to $\mathbb{Z}_2^{\mathsf{k}-1}$.

Recall that we write $\hat{M}$ for the space of equivalence classes of points in the universal covering of M that are not separable by continuous CR-functions. Our first main result now is:

4.3. Theorem. *In case $a \in E$ is an invertible matrix, every continuous* CR*-function on the universal covering $\tilde{M}$ of the orbit $M = G(a)$ is the pullback of some* CR*-function on M, that is, $\hat{M} = M$. In particular, every non-trivial covering of M is a non-embeddable* CR*-manifold.*

Proof. $S := \{z \in E : \det(z) = \det(a)\}$ is a G-invariant complex submanifold of E containing the orbit M. For every $j \leqq p$ consider the real valued function $\mu_j := \sigma_1 \sigma_2 \cdots \sigma_j$ on E. Then

$$Y := \{z \in S : \mu_j(z) < \mu_j(a) \text{ for all } j < p\}$$

is a (possibly empty) G-invariant domain in S satisfying condition 3.2(i) as a consequence of [15], Theorem 12.1. But also 3.2(ii) holds since, for every $y \in S$, the orbit $G \cdot y$ is a generic CR-submanifold of S, compare [15], section 8. Because of Proposition 3.2 therefore we only have to verify 3.2(iii), that is, that $M \cup Y$ is simply-connected. But this follows with the odd functional calculus on E that we briefly recall here (compare the discussion between 10.7 and 10.8 in [15]): Every odd function $f : \mathbb{R} \to \mathbb{R}$ induces a G-equivariant mapping $\boldsymbol{f} : E \to E$ with the property that for every real diagonal matrix $d = (d_{ij}) \in E$ the image $\boldsymbol{f}(d)$ is the real diagonal matrix $\big(f(d_{ij})\big)$ in E. Without loss of generality we may assume that a is the real diagonal matrix with diagonal entries $a_i = \sigma_i(a)$. Let $c \in E$ be the unit matrix multiplied with the real factor $\beta := \mu_p(a)^{1/p} > 0$. The orbit $G \cdot c$ is simply-connected and contained in $M \cup Y$. For every $0 \leqq s \leqq 1$ let f_s be the odd function on $\mathbb{R}$ satisfying $f_s(t) = \beta^{1-s}t^s$ for all $t > 0$. Then the family of all f_s induces a continuous retraction from $M \cup Y$ onto the simply-connected orbit $G \cdot c$, i.e. 3.2(iii) holds and the claim follows from Proposition 3.2. □

4.4. Corollary. *For all $a, b \in E$ with a invertible, the following conditions are equivalent*:

(i) *The universal covering spaces of the orbits $G \cdot a$ and $G \cdot b$ are* CR*-isomorphic.*

(ii) *There exist coverings $U \to G \cdot a$, $V \to G \cdot b$ such that U, V are* CR*-isomorphic.*

(iii) *The orbits $G \cdot a$ and $G \cdot b$ are* CR*-isomorphic.*

Proof. Every CR-isomorphism $U \to V$ maps maximal sets that cannot be separated by continuous CR-functions, to sets with the same property and hence induces a CR-isomorphism of the corresponding orbits as a consequence of 4.3. □

Let as before $k = k(a)$ be the maximal number of different singular values of $a \in E$. Then the orbit $M = G \cdot a$ has pointed fundamental group $\pi_1(M, a) \simeq K/K^0 \simeq \mathbb{Z}_2^{k-1}$. Since the coverings of M are in 1:1-correspondence to the subgroups of $\pi_1(M, a)$ we count, for instance, $2^{k-1} - 1$ different coverings of degree 2 and $\dfrac{1}{3}(2^{k-1} - 1)(2^{k-2} - 1)$ different coverings of degree 4 for M. The question arises, which of these covering spaces are isomorphic as CR-manifolds. For this recall from (2.5) the definition of the involution $\theta : \Delta^+ \to \Delta^+$ and of the CR-diffeomorphism $\theta_a : G \cdot a \to G \cdot \big(\theta(a)\big)$. For every $a \in \mathsf{Fix}(\theta)$ the group $\Theta_a := \{\mathsf{id}, \theta_a\}$ of order $\leqq 2$ acts in a natural way on the fundamental group $\pi_1(M, a)$ and hence on the set of coverings of M. Since for every invertible $a \in E$ we may assume $a \in \Delta^+$ without loss of generality, the following proposition together with 4.4 solves the CR-equivalence problem for covering spaces of invertible G-orbits in E.

4.5. Proposition. *For every $a \in \Delta^+$ with $a \neq \mathbb{1}$ we have for the orbit $M = G \cdot a$*:

(i) *In case $a \in \mathsf{Fix}(\theta)$ two covering spaces of M are* CR*-equivalent if and only if they are equivalent under the action of the group Θ_a.*

(ii) *In case $a \notin \mathsf{Fix}(\theta)$ the covering spaces of M are pairwise* CR*-inequivalent.*

Proof. Let $\Gamma_j \subset \pi_1(M, a)$ be subgroups with associated coverings $\tau_j : N_j \to M$ for $j = 1, 2$ and assume that $\xi : N_1 \to N_2$ is a CR-homeomorphism. Then ξ lifts to a transformation $\tilde{\xi} \in \mathsf{Aut}_{\mathrm{CR}}(\tilde{M})$ of the universal covering $\tilde{M}$ of M. As a consequence of 4.4, $\tilde{\xi}$ is the lifting of a transformation in $\mathsf{Aut}_{\mathrm{CR}}(M)$. In case $a \in \mathsf{Fix}(\theta)$ the group $\mathsf{Aut}_{\mathrm{CR}}(M)$ is generated by its connected identity component and θ_a [15], Corollary 13.6, proving (i). Again by [15], Corollary 13.6, $\mathsf{Aut}_{\mathrm{CR}}(M)$ is connected if $a \in \Delta^+$ is not in $\mathsf{Fix}[\theta]$, proving (ii). □

5. Orbits of non-invertible matrices

Having settled the invertible matrix case in the preceding section let us assume for the rest of this section that $a \neq 0$ in E is not invertible and hence has rank r with $0 < r < p$. Denote by $E_k \subset E$ for every $0 \leqq k \leqq p$ the (locally closed) complex submanifold consisting of all matrices with rank k in E. Then the orbit

$$M := G \cdot a = \{z \in E_r : \sigma_j(z) = \sigma_j(a) \text{ for all } j \leqq r\}$$

is a generic and minimal CR-submanifold of E_r. In contrast to the case of invertible matrices, M is circular, i.e., invariant under all transformations $z \mapsto e^{it}z$, $t \in \mathbb{R}$.

The group $\mathsf{GL}(p, \mathbb{C})$ acts by matrix multiplication from the left on $\mathbb{C}^{p \times r}$. Also E is a linear $\mathsf{GL}(p, \mathbb{C})$-space with respect to $g \cdot z = gzg'$ for all $g \in \mathsf{GL}(p, \mathbb{C})$, and the holomorphic mapping

$$(5.1) \qquad \varphi : \mathbb{C}^{p \times r} \to E, \quad \varphi(z) := zz',$$

is $\mathsf{GL}(p, \mathbb{C})$-equivariant and has the closure

$$R := E_r \cup E_{r-1} \cup \cdots \cup E_0 \quad \text{of} \quad E_r \text{ in } E$$

as image. The complex orthogonal group $\mathsf{O}(r, \mathbb{C})$ acts on $\mathbb{C}^{p \times r}$ from the right and the categorical quotient $\mathbb{C}^{p \times r} /\!\!/ \mathsf{O}(r, \mathbb{C})$ is a normal complex space. The mapping φ is $\mathsf{O}(r, \mathbb{C})$-invariant and induces a biholomorphic map from $\mathbb{C}^{p \times r} /\!\!/ \mathsf{O}(r, \mathbb{C})$ onto the complex analytic cone $R = \bar{E}_r$ in E, compare e.g. [11], p. 182.

Without loss of generality we may assume $a \in \Delta^0$, see (2.3). Define $\lambda_1 > \lambda_2 > \cdots > \lambda_k = 0$ with multiplicities $r_1, \ldots, r_k$ as in (4.1) and identify M as homogeneous space with G/K, where the isotropy subgroup K at a is given by (4.2). The subgroup

$$\hat{K} := \mathsf{S}\big(\mathsf{O}(r_1) \times \mathsf{O}(r_2) \times \cdots \times \mathsf{O}(r_{k-1})\big) \times \mathsf{SO}(r_k) \subset K$$

has index 2 in K. Therefore, the homogeneous space $G/\hat{K}$ is a 2-sheeted covering of $M = G/K$.

Recall from Sect. 3 the definition of the covering $\hat{M} \to M$, which in a way is maximal with respect to the property that the covering space is embeddable. Our second main result now states:

5.2. Theorem. *For every orbit $M = G \cdot a$ with $a \neq 0$ non-invertible in E, the covering $\hat{M} \to M$ is* 2:1 *and* CR*-isomorphic to $G/\hat{K} \to G/K$.*

Proof. In a first step we show that M admits a 2-sheeted covering $N \to M$ with N an embeddable (connected) CR-manifold. For $r = \operatorname{rank}(a)$ and φ as in (5.1) fix a matrix $c \in \varphi^{-1}(a)$ and consider the pre-image

$$(5.3) \qquad S := \varphi^{-1}(M) = \{ucv : u \in \mathsf{SU}(p), v \in \mathsf{O}(r, \mathbb{C})\},$$

which is a generic CR-submanifold of $\mathbb{C}^{p\times r}$. We claim that S is connected. Since $\mathsf{SU}(p)$ and $\mathsf{SO}(r,\mathbb{C})$ are connected, it is enough to show $cv \in S$ for some $v \in \mathsf{O}(r,\mathbb{C})$ with $\det(v) = -1$. Without loss of generality we may assume that the matrix $c \in \mathbb{C}^{p\times r}$ is diagonal, that is, $c_{jk} = 0$ if $j \neq k$. But then, if we also choose v to be diagonal, there is a matrix $w \in \mathsf{O}(p-r)$ with $u := \begin{pmatrix} v & 0 \\ 0 & w \end{pmatrix} \in \mathsf{SU}(p)$, and $cv = uc \in S$ proves the claim. The differential of $\varphi : S \to M$ at $c \in S$ induces a complex linear surjection $H_cS \to H_aM$ of the corresponding holomorphic tangent spaces. Consequently, M can be identified as CR-manifold with $S/\mathsf{O}(r,\mathbb{C})$, and $S/\mathsf{SO}(r,\mathbb{C})$ is a 2-sheeted covering of M. Since every $r\times r$-minor on $\mathbb{C}^{r\times p}$ is an $\mathsf{SO}(r,\mathbb{C})$-invariant holomorphic function that is not $\mathsf{O}(r,\mathbb{C})$-invariant, the quotient CR-manifold $S/\mathsf{SO}(r,\mathbb{C})$ is separable by CR-functions and hence embeddable by Proposition 3.1.

In a next step we show that every covering $N \to M$ with N embeddable has degree $\leqq 2$. For this consider the G-invariant domain

$$Y := \{z \in E_r : \sigma_j(z) < \sigma_j(a) \text{ for all } j \leqq r\}$$

in the complex submanifold $E_r \subset E$. Then property 3.2(i) is satisfied as a consequence of [15], Theorem 12.1. But also 3.2(ii) holds since, for every $y \in Y$, the orbit $G\cdot y$ is a generic CR-submanifold of Y, compare [15], section 8. Let $e \in E_r$ be a matrix with $\sigma_1(e) = \sigma_r(e) < \sigma_r(a)$. Then the orbit $G\cdot e$ has fundamental group $\mathbb{Z}_2$ and is contained in $M \cup Y$. As in the proof of Theorem 3.2 we conclude that $G\cdot e$ is a retract of $M\cup Y$. Proposition 3.2 now can be applied and gives that the covering $N \to M$ has degree $\leqq 2$. Both steps together complete the proof. □

As a consequence of Proposition 3.1 the 2-sheeted covering space $\hat{M}$ of M can be realized as a G-orbit in some linear G-space. In the following we give such a realization. For this identify $\mathbb{C}^p = \mathbb{C}^{p\times 1}$ and consider the r-fold exterior product $F := \Lambda^r(\mathbb{C}^p)$. Then F is also a linear $\mathsf{GL}(p,\mathbb{C})$-space if we put $g\cdot\omega := \Lambda^r(g)(\omega)$ for all $g \in \mathsf{GL}(p,\mathbb{C})$ and $\omega \in F$. In analogy to (5.1) consider the $\mathsf{GL}(p,\mathbb{C})$-equivariant holomorphic mapping

$$\psi : \mathbb{C}^{p\times r} \to F, \quad \psi(z) := z_1 \wedge z_2 \wedge \cdots \wedge z_r, \tag{5.4}$$

where $z_1,\dots,z_r \in \mathbb{C}^p$ are the column vectors of the matrix $z = (z_1,\dots,z_r)$. It is easily seen that $\psi(zv) = \det(v)\psi(z)$ holds for all $(z,v) \in \mathbb{C}^{p\times r}\times \mathsf{GL}(r,\mathbb{C})$ and hence that ψ is $\mathsf{SO}(r,\mathbb{C})$-invariant but not $\mathsf{O}(r,\mathbb{C})$-invariant. Consider $E\times F$ as direct product of linear $\mathsf{GL}(p,\mathbb{C})$-spaces. Then the image Q of the algebraic map $\chi := (\varphi,\psi) : \mathbb{C}^{p\times r} \to E\times F$ is a $\mathsf{GL}(p,\mathbb{C})$-invariant complex analytic subset of $E\times F$. In addition, Q is invariant under the biholomorphic transformation ε given by $(x,\omega) \mapsto (x,-\omega)$. The quotient $\mathbb{C}^{p\times r}/\!\!/\mathsf{SO}(r,\mathbb{C})$ is a normal complex space and χ induces a holomorphic homeomorphism $\mathbb{C}^{p\times r}/\!\!/\mathsf{SO}(r,\mathbb{C}) \to Q$. Actually, this homeomorphism is biholomorphic by [6], and hence Q is a normal Stein space.

Denote by π the restriction to Q of the canonical projection $E\times F \to E$. The preimage $\pi^{-1}(E_r)$ is a simply-connected domain in Q and $\pi : \pi^{-1}(E_r) \to E_r$ is a 2-sheeted covering of complex manifolds. For every $k < r$ the holomorphic mapping $\pi : \pi^{-1}(E_k) \to E_k$ is bijective.

Now fix a matrix $c \in \mathbb{C}^{p\times r}$ with $a = \varphi(c)$ and put $\alpha := \psi(c)$. For S, defined in (5.3), then the orbit $\chi(S) = G \cdot (a, \alpha)$ in Q is a 2-sheeted covering of M with respect to π and hence can be identified as CR-manifold with the covering $\hat{M} = S/\mathsf{SO}(r)$ of M.

Denote by $\mathscr{C}_{\mathrm{CR}}(M)$ the complex Banach algebra of all continuous CR-functions on M. Then by [15], every $f \in \mathscr{C}_{\mathrm{CR}}(M)$ has a unique continuous extension to the compact subset

$$\mathscr{X}(a) := \{z \in R : \sigma_j(z) \leqq \sigma_j(a) \text{ for all } j \leqq r\}$$

of $R = \bar{E}_{(r)}$, whose restriction to the domain

$$\mathscr{Y}(a) := \{z \in R : \sigma_j(z) < \sigma_j(a) \text{ for all } j \leqq r\}$$

in the complex Stein space R is holomorphic. Actually, via point evaluation, the spectrum of $\mathscr{C}_{\mathrm{CR}}(M)$ identifies with the set $\mathscr{X}(a)$. As a consequence, the spectrum of $\mathscr{C}_{\mathrm{CR}}(\hat{M})$ can be identified with the compact subset $\hat{\mathscr{X}}(a) := \pi^{-1}\big(\mathscr{X}(a)\big)$ of Q, and every $f \in \mathscr{C}_{\mathrm{CR}}(\hat{M})$ has a unique continuous extension to $\hat{\mathscr{X}}(a)$ whose restriction to the domain $\hat{\mathscr{Y}}(a) := \pi^{-1}\big(\mathscr{Y}(a)\big)$ in the Stein space Q is holomorphic. Indeed, ε splits $\mathscr{C}_{\mathrm{CR}}(\hat{M})$ into $+1$- and -1-eigenspace, and every f in the -1-eigenspace is a square root of a function in the $+1$-eigenspace.

6. Final remarks

In this final section $a \in E$ is an arbitrary element, may be invertible or not. The following remark is easily seen, compare also [15].

6.1. Remark. For every $a \in E$ the following conditions are equivalent:

(i) The orbit $G \cdot a$ is simply-connected.

(ii) The orbit $G \cdot a$ is totally real in E.

(iii) All singular values of a coincide.

(iv) $aa^* = s\mathbb{1}$ for some $s \geqq 0$.

6.2. Proposition. *For every $a \in E$ the following conditions are equivalent*:

(i) *The universal covering of the orbit $G \cdot a$ is embeddable as* CR*-manifold.*

(ii) *All non-zero singular values of a coincide.*

(iii) $aa^*a = sa$ *for some* $s \geqq 0$.

Proof. (i) $\Rightarrow$ (ii). Suppose that (i) holds. Because of Remark 6.1 we may assume without loss of generality that $G \cdot a$ is not simply-connected. As a consequence of Theorem 4.3 the matrix a is not invertible and hence has 0 as singular value. By Theorem 5.2 the orbit $G \cdot a$ has fundamental group $\mathbb{Z}_2$ and hence precisely 2 different singular values.

(ii) $\Rightarrow$ (iii). Suppose that (ii) holds. The subgroup $\{\lambda g : \lambda \in \mathbb{C}^*, g \in G\}$ of $\mathsf{GL}(E)$ maps G-orbits onto G-orbits and respects all conditions (i)–(iii). We may therefore assume $a \in \Delta$ without loss of generality, see (2.2). But then $aa^*a = a$ implies (iii).

(iii) $\Rightarrow$ (i). Suppose that (iii) holds. We may assume $s > 0$ since otherwise $a = 0$ and (i) would hold. But then we even may assume $a \in \Delta$ and $s = 1$, that is, 1 is the only non-zero singular value of a. In case a is invertible, Remark 6.1 gives that $G \cdot a$ is simply-connected. In case a is not invertible, the universal covering of $M = G \cdot a$ is the only 2-sheeted covering of M and hence coincides with $\hat{M}$ by Theorem 5.2. In any case, (i) must be true. □

The set of all $a \in E$ for which all singular values are pairwise different, i.e. $\mathsf{k}(\mathsf{a}) = \mathsf{p}$, is open and dense in E. For every such a the orbit $M = G \cdot a$ has CR-codimension $p - 1$ and the isotropy subgroup $K = \{g \in G : g \cdot a = a\}$ is isomorphic to $\mathbb{Z}_2^{p-1}$. As a consequence, the universal covering $\tilde{M}$ of M can be identified as homogeneous G-space with G acting on itself by left translations. In particular, for $p = 2$ and $0 \leqq t < 1$ the universal covering $\tilde{M}_t$ of the orbit $M_t := G \cdot \begin{pmatrix} 1 & 0 \\ 0 & t \end{pmatrix}$ is of hypersurface type and gives a leftinvariant strongly pseudoconvex CR-structure on $\mathsf{SU}(2) \simeq S^3$. In addition, the $\tilde{M}_t$, $0 \leqq t < 1$, are pairwise inequivalent as CR-manifolds and also are non-embeddable except for $t = 0$ ($\tilde{M}_0$ is CR-equivalent to the standard embedding of the 3-sphere S^3 in $\mathbb{C}^2$), compare [16], [9], [3], [13].

References

[1] *Andreotti, A., Fredricks, G. A.*, Embeddability of real analytic Cauchy-Riemann manifolds, Ann. Scu. Norm. Sup. Pisa Cl. Sci. (4) **6** (1979), 285–304.

[2] *Andreotti, A., Hill, C. D.*, Complex characteristic coordinates and tangential Cauchy-Riemann equations, Ann. Scu. Norm. Sup. Pisa (3) **26** (1972), 299–324.

[3] *Azad, H., Huckleberry, A., Richthofer, W.*, Homogeneous CR-manifolds, J. reine angew. Math. **358** (1985), 125–154.

[4] *Boutet de Monvel, L.*, Intégration des équations de Cauchy-Riemann induites formelles, Séminaire Goulaouic-Lions-Schwartz 1974–1975; Équations aux derivées partielles linéaires et non lináires, pp. Exp. No. 9, 14 pp.

[5] *Bröcker, T., tom Dieck, T.*, Representations of Compact Lie Groups, Springer, Berlin-Heidelberg-New York 1985.

[6] *Bruns, W.*, private communication.

[7] *Carlson, J. A., Hill, C. D.*, On the maximum modulus principle for the tangential Cauchy-Riemann equations, Math. Ann. **208** (1974), 91–97.

[8] *Ellis, D., Hill, C. D., Seabury, C. C.*, The maximum modulus principle, I, Necessary conditions, Indiana Univ. Math. J. **25** (1976), 709–715.

[9] *Falbel, E.*, Non-embeddable CR-manifolds and surface singularities, Invent. math. **108** (1992), 49–65.

[10] *Forstnerič, F.*, Mappings of strongly pseudoconvex Cauchy-Riemann manifolds, Several complex variables and complex geometry, Part 1 (Santa Cruz, CA, 1989), Proc. Sympos. Pure Math. **52**, Part 1, Amer. Math. Soc., Providence, RI (1991), 59–92.

[11] *Goodman, R., Wallach, N. R.*, Representations and invariants of the classical groups, Encycl. Math. Appl. **68**, Cambridge University Press, Cambridge 1998.

[12] *Hill, C. D., Nacinovich, M.*, A weak pseudoconcavity condition for abstract almost CR manifolds, Invent. math. **142** (2000), 251–283.

[13] *Huckleberry, A., Richthofer, W.*, Recent Developments in homogneous CR-hypersurfaces, Contributions to several complex variables, Aspects Math. E9, Vieweg (1986), 149–177.

[14] *Iordan, A.*, The maximum modulus principle for CR functions, Proc. Amer. Math. Soc. **96** (1986), 465–469.

[15] *Kaup, W., Zaitsev, D.*, On the CR-structure of compact group orbits associated with bounded symmetric domains, Invent. math. **153** (2003), 45–104.

[16] *Rossi, H.*, Attaching analytic spaces to an analytic space along a pseudoconcave boundary, Proc. Conf. Complex Analysis (Minneapolis 1964), Springer, Berlin (1965), 242–256.

[17] *Tumanov, A. E.*, Extremal discs and the regularity of CR mappings in higher codimension, Amer. J. Math. **123** (2001), 445–473.
[18] *Tumanov, A. E., Henkin, G. M.*, Local characterization of holomorphic automorphisms of Siegel domains, Funktsional. Anal. i Prilozhen. **17** (1983), 49–61.

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