November 2, 2007 Lecturer Dmitri Zaitsev Michaelmas Term 2007

Course 414 2007-08

S h e e t 2

Due: after the lecture next Monday

Exercise 1

Sketch the set of points give by the condition:

(i) $1 < |z| < 2$;

(ii) $1 < |z+i| < 2$;

(iii) $\mathsf{Re}((1-i)\bar{z}) \geq 2$.

Exercise 2

Let $f \colon \Omega \to \mathbb{C}$ be holomorphic. Define the new function $\bar{f}$ by $\bar{f}(z) := \overline{f(\bar{z})}$. Show that $\bar{f}$ is holomorhic on the open set $\bar{\Omega} := \{\bar{z} : z \in \Omega\}$.

Exercise 3

Using the Cauchy-Riemann equations, show:

(i) if a holomorphic function f satisfies $\mathsf{Re} f = \text{const}$, then $f = \text{const}$.

(ii) if $f = u + iv$ is holomorphic and $a, b \in \mathbb{C} \setminus \{0\}$ are such that $au + bv = \text{const}$, then again $f = \text{const}$.

Exercise 4

Give an example of a maximal open set $\Omega \subset \mathbb{C}$, where the given multiple-valued function has a holomorphic branch:

(i) $z^{1/3}$;

(ii) $\mathsf{log}(z+1)$;

(iii) $\sqrt{e^z}$.

Exercise 5

Let f be holomorphic in an open set $\Omega \subset \mathbb{C}$ and $\Phi \colon [0,1] \times [a,b] \to \Omega$ be a "homotopy of closed arcs", i.e. a continuous map such that $\Phi(t,a) = \Phi(t,b)$ for all $t \in [0,1]$. Show that $\int_{\gamma_0} f dz = \int_{\gamma_1} f dz$ where $\gamma_t(\lambda) := \Phi(t,\lambda)$. (Hint. Reduce to Cauchy's theorem for arcs homotopic with fixed endpoints.)