Course 414 2005-06

Sheet 6

Due: after the lecture beginning the next term

Exercise 1

Determine the type of singularity (removable, pole, essential or not isolated) and calculate the residue in case it is isolated:

(i) $f(z) = \frac{\sin z}{z-\pi}$ at $z_0 = \pi$;

(ii) $f(z) = \frac{z}{\cos z - 1}$ at $z_0 = 0$;

(iii) $f(z) = z^2 e^{-1/z^3}$ at $z_0 = 0$;

(iv) $f(z) = \frac{z^2}{e^{1/z}-1}$ at $z_0 = 0$;

Exercise 2

For each function f from the previous exercise, determine a maximal open set $\Omega \subset \mathbb{C}$ such that f is meromorphic in Ω.

Exercise 3

Let f have a pole of order m at a point z_0 and g have pole of order n at the same point.

(i) Does $f+g$ always have an isolated singularity at z_0?

(ii) Does $f+g$ always have a pole at z_0?

(iii) Same question for $h = fg$?

(iv) In cases $f+g$ or fg have a pole at z_0, what are the possible pole orders?

Exercise 4

If f and g are entire functions (holomorphic in $\mathbb{C}$) and $|f(z)| \leq |g(z)|$ for all z, show that $f(z) = cg(z)$ for some constant c.

Exercise 5

Suppose that f is meromorphic in $\mathbb{C}$ and bounded outside a disk $B_R(0)$. Show that f is rational.

Exercise 6

Evaluate the integrals:

(i) $\int_{|z|=3} z^{-1}(z-1)^{-1}(z-2)^{-1}\,dz$;

(ii) $\int_0^{2\pi}(a+b\mathsf{cos}\theta)^{-1}\mathsf{cos}\theta d\theta$;

(iii) $\int_{-\infty}^{\infty}(x^2+\pi^2)^{-2}\mathsf{cos}x dx$;

(iii) $\int_{-\infty}^{\infty}(x-i\pi)^{-2}e^{ix}dx$;

(iv) $\int_0^{\infty}\frac{dx}{\sqrt{x}(x^2+1)}$;

Exercise 7

Use Rouché's theorem to find the number of zeroes of the polynomial inside the circle $|z|=1$:

(i) $z^6-5z^4+z^3-2z$;

(ii) $2z^4-2z^3+z^2-z+7$;