Course 2E1 2006-07 (SF Engineers & MSISS & MEMS)

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Due: at the end of the tutorial

Exercise 1

Find the Fourier series of the function $f(x)$:

(iii) $f(x) = \begin{cases} 2x - 1 \text{ if } -\pi < x < 0 \\ 1 - 2x \text{ if } 0 < x < \pi \end{cases}.$

Solution

First split $f(x)$ into the sum of its even and odd parts:

$$f(x) = g(x) + h(x), \quad g(x) = \frac{f(x) + f(-x)}{2}, \quad h(x) = \frac{f(x) - f(-x)}{2}.$$

Then

$$g(x) = \begin{cases} 2x \text{ if } -\pi < x < 0 \\ -2x \text{ if } 0 < x < \pi \end{cases}, \quad h(x) = \begin{cases} -1 \text{ if } -\pi < x < 0 \\ 1 \text{ if } 0 < x < \pi \end{cases}.$$

Now for $g(x)$, we only need to calculate its even expansion, i.e. a_0 and a_n:

$$a_0 = \frac{1}{2\pi} \int_{-\pi}^{\pi} g(x)\, dx = \frac{1}{2\pi} \left(\int_{-\pi}^{0} (-2x)\, dx + \int_{0}^{\pi} 2x\, dx \right)$$

$$= \frac{1}{2\pi} \left((-x^2)\Big|_{-\pi}^{0} + x^2\Big|_{0}^{\pi} \right) = \frac{1}{2\pi} \left(\pi^2 + \pi^2 \right) = \pi.$$

$$a_n = \frac{1}{\pi} \int_{-\pi}^{\pi} g(x) \cos nx\, dx = \frac{1}{2\pi} \left(\int_{-\pi}^{0} (-2x) \cos nx\, dx + \int_{0}^{\pi} 2x \cos nx\, dx \right).$$

In the first integral we change the variable $y = -x$:

$$\int_{-\pi}^{0} (-2x) \cos nx\, dx = \int_{\pi}^{0} (2y) \cos ny\, d(-y) =$$

$$\int_{0}^{\pi} (2y) \cos ny\, dy = \int_{0}^{\pi} (2x) \cos nx\, dx.$$

Then

$$a_n = \frac{1}{2\pi} 2 \int_0^\pi 2x \,\mathsf{cos} nx \, dx = \frac{2}{\pi} \int_0^\pi x \,\mathsf{cos} nx \, dx.$$

To integrate by parts, we write $x \,\mathsf{cos} nx = uv'$ with $u = x$ and $v' = \mathsf{cos} nx$, whence $v = \frac{\mathsf{sin} nx}{n}$. Then

$$a_n = \frac{2}{\pi} \left((uv)|_0^\pi - \int_0^\pi u'v \, dx \right) = \frac{2}{\pi} \left(\frac{x \mathsf{sin} nx}{n} \bigg|_0^\pi - \int_0^\pi \frac{\mathsf{sin} nx}{n} \, dx \right) = \frac{2}{\pi} \left. \frac{\mathsf{cos} nx}{n^2} \right|_0^\pi$$

$$= \frac{2(\mathsf{cos} n\pi - 1)}{\pi n^2} = \frac{2((-1)^n - 1)}{\pi n^2}.$$

Finally, for the odd function $h(x)$, we only have the odd expansion:

$$b_n = \frac{1}{\pi} \int_{-\pi}^{\pi} h(x) \,\mathsf{sin} nx \, dx = \frac{1}{\pi} \left(\int_{-\pi}^{0} \mathsf{sin} nx \, dx + \int_0^\pi -\mathsf{sin} nx \, dx \right) = -\frac{2}{\pi} \int_0^\pi \mathsf{sin} nx \, dx$$

$$= \frac{2}{\pi} \left. \frac{\mathsf{cos} nx}{n} \right|_0^\pi = \frac{2}{\pi} \frac{(-1)^n - 1}{n}.$$

To find the series for f we add those for g and h:

$$a_0 + \sum_{n=1}^{\infty} (a_n \mathsf{cos} nx + b_n \mathsf{sin} nx) = \pi + \sum_{n=1}^{\infty} \left(\frac{2((-1)^n - 1)}{\pi n^2} \mathsf{cos} nx + \frac{2((-1)^n - 1)}{\pi n} \mathsf{sin} nx \right).$$