Course 2E1 2004-05 (SF Engineers & MSISS & MEMS)

Sheet 9

Due: in the tutorial sessions next Wednesday/Thursday

Exercise 1

Find the volume of the solid S:

(i) S is the pyramid with vertices $(0,0,0)$, $(0,1,0)$, $(2,0,0)$, $(0,0,3)$;

Solution. S is bounded by the planes $x=0$, $y=0$, $z=0$ and the plane passing through the points $(0,1,0)$, $(2,0,0)$, $(0,0,3)$. The latter has general equation $ax+by+cz=d$ with coefficients a,b,c,d to be found. Substitution of the points $(0,1,0)$, $(2,0,0)$, $(0,0,3)$ for (x,y,z) yields the system

$$b=d, \quad 2a=d, \quad 3c=d.$$

It is enough to find one solution. With $d=6$ we get $a=3$, $b=6$, $c=2$ and the equation is

$$3x+6y+2z=6.$$

Projection to the x-axis gives $0 \le x \le 2$ as limits for x. Projection to the xy-plane is the triangle $x \ge 0$, $y \ge 0$, $x+2y \le 2$. Hence the limits for y are $0 \le y \le 1-x/2$ (for fixed x). Finally, the limits for z (for fixed x,y) are $0 \le z \le 3-3y-3x/2$, the upper limit is obtained from solving the above equation for z. The volume is now

$$V = \int_0^2 \int_0^{1-x/2} \int_0^{3-3y-3x/2} dz\,dy\,dx = \int_0^2 \int_0^{1-x/2} (3-3y-3x/2)\,dy\,dx$$

$$= \int_0^2 (3y-3y^2/2-3xy/2)\Big|_{y=0}^{y=1-x/2} dx = 1.$$

(ii) S is given by $x \ge 0$, $1 \le y \le 2$, $z \ge 0$, $z \le 4-x^2$;

Solution. $z \ge 0$, $z \le 4-x^2$ together yield $-2 \le x \le 2$, with $x \ge 0$ we have the limits $0 \le x \le 2$. S is the solid above the rectangular $0 \le x \le 2$, $1 \le y \le 2$ and below the surface $z = 4-x^2$.

$$V = \int_0^2 \int_1^2 \int_0^{4-x^2} dz\,dy\,dx = 8-8/3.$$

(iii) S is given by $x \geq 0$, $y \geq 0$, $x + y \leq 1$, $0 \leq z \leq x^2 + y^2$.

Solution. Projection to the xy-plane is given by $x \geq 0$, $y \geq 0$, $x + y \leq 1$ and so x-limits are $0 \leq x \leq 1$ and y-limits for fixed x are $0 \leq y \leq 1 - x$. Finally, z-limits for fixed x, y are given: $0 \leq z \leq x^2 + y^2$. S is the solid above the triangular $x \geq 0$, $y \geq 0$, $x + y \leq 1$ in the xy-plane and below the surface $z = x^2 + y^2$.

$$V = \int_0^1 \int_0^{1-x} \int_0^{x^2+y^2} dz\, dy\, dx = 1/6.$$

Exercise 2

Find area and center of mass (assuming constant density $\delta = 1$) of the bounded region R:

(i) R is given by $2 \leq x \leq 4$, $3 \leq y \leq 7$;

Solution. The area is

$$A = \int_2^4 \int_3^7 dy\, dx = 8,$$

the mass is

$$M = \int_2^4 \int_3^7 \delta(x, y)\, dy\, dx = 8,$$

the moments are

$$M_1 = \int_2^4 \int_3^7 x\delta(x, y)\, dy\, dx = \int_2^4 \int_3^7 x\, dy\, dx$$

and

$$M_2 = \int_2^4 \int_3^7 y\delta(x, y)\, dy\, dx = \int_2^4 \int_3^7 y\, dy\, dx.$$

Then the center of mass is the point with coordinates $(\frac{M_1}{M}, \frac{M_2}{M})$. The evaluation is simple and is omitted, also in what follows.

(ii) R is given by $x \geq 0$, $y \geq 0$, $2x + y \leq 2$;

Solution. The limits for x are found by eliminating it from the given inequalities. We rewrite the last inequality as $x \leq 1 - y/2$ and use the second to obtain $x \leq 1$. Hence $0 \leq x \leq 1$ and the last two inequalities give $0 \leq y \leq 2 - 2x$. Now the area is

$$A = \int_0^1 \int_0^{2-2x} dy\, dx,$$

the mass is

$$M = \int_0^1 \int_0^{2-2x} dy\, dx$$

and the moments are

$$M_1 = \int_0^1 \int_0^{2-2x} x\, dy\, dx$$

and

$$M_2 = \int_0^1 \int_0^{2-2x} y\, dy\, dx$$

and the center of mass has the coordinates $(\frac{M_1}{M}, \frac{M_2}{M})$.

(iii) R is given by $y \geq 0$, $y \leq 1 - x^2$.

Solution. Here, eliminating y, we have $0 \leq 1 - x^2$ which yields $-1 \leq x \leq 1$. Hence the area is

$$A = \int_{-1}^1 \int_0^{1-x^2} dy\, dx,$$

the mass is

$$M = \int_{-1}^1 \int_0^{1-x^2} dy\, dx$$

and the moments are

$$M_1 = \int_{-1}^1 \int_0^{1-x^2} x\, dy\, dx$$

and

$$M_2 = \int_{-1}^1 \int_0^{1-x^2} y\, dy\, dx$$

and the center of mass has the coordinates $(\frac{M_1}{M}, \frac{M_2}{M})$.