

Course 2E1 2004-05 (SF Engineers & MSISS & MEMS)

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Due: in the tutorial sessions first Wednesday/Thursday in the next term

Exercise 1

Find bases and dimensions for the row and the column space of the matrix:

(i) $\begin{pmatrix} 5 & -1 \end{pmatrix}$;

Solution. Dividing the 1st row by 5 we get the row echelon form: $\begin{pmatrix} 1 & -1/5 \end{pmatrix}$ with one row and one leading 1. A basis for the row space is the set of rows with leading 1, so it is $\{(1, -1/5)\}$. A basis for the column space is the set of columns in the original matrix having the same positions as the leading 1's in the row equivalent matrix. The position is the 1st, so we take the 1st column in the original matrix, which is the 1-vector 5. The basis is $\{(5)\}$. The dimension in both cases is 1 (it is always the same for the row and the column space).

(ii) $\begin{pmatrix} 5 \\ -1 \end{pmatrix}$;

Solution. To get the row echelon form, we divide the 1st row by 5 and then add it to the 2nd row: $\begin{pmatrix} 1 \\ 0 \end{pmatrix}$. Following the same rule, a basis for the row space is $\{(1)\}$ and for the column space is $\begin{pmatrix} 5 \\ -1 \end{pmatrix}$, the dimension is 1.

(iii) $\begin{pmatrix} -1 & 2 \\ 1 & -2 \end{pmatrix}$;

Solution. The matrix is row equivalent to $\begin{pmatrix} 1 & -2 \\ 0 & 0 \end{pmatrix}$, hence a basis for the row space is $\{(1, -2)\}$ and for the column space $\{(-1, 1)\}$ (the column having position 1). Both dimensions are 1.

$$(iv) \begin{pmatrix} -1 & 2 & 0 \\ 1 & -2 & 1 \end{pmatrix};$$

Solution. The matrix is row equivalent to $\begin{pmatrix} 1 & -2 & 0 \\ 0 & 0 & 1 \end{pmatrix}$, hence a basis for the row space is $\{(1, -2, 0), (0, 0, 1)\}$ (both rows). Furthermore, the leading 1's are at the positions 1 and 3, hence a basis for the column space is given by the columns in the original matrix at the same positions: $\{(-1, 1), (0, 1)\}$, both dimensions are 2.

$$(v) \begin{pmatrix} -1 & 2 \\ 1 & -2 \\ 1 & 0 \end{pmatrix};$$

Solution. The matrix is row equivalent to $\begin{pmatrix} 1 & -2 \\ 0 & 1 \\ 0 & 0 \end{pmatrix}$, hence a basis for the row space is $\{(1, -2), (0, 1)\}$ and for the column space $\{(-1, 1, 1), (2, -2, 0)\}$, both dimensions are 2.

Exercise 2

Find the rank and the nullity of the matrices in Exercise 1.

Solution. The rank is by definition the dimension of both row and column spaces (always equal) and the nullity is the dimension of the null space (and equal to the number of the free variables in the associated system) which can be calculated as the number of the columns minus the rank. So we have:

- (i) The rank is 1 and the nullity is $2 - 1 = 1$.
- (ii) The rank is 1 and the nullity is $1 - 1 = 0$.
- (iii) The rank is 1 and the nullity is $2 - 1 = 1$.
- (iv) The rank is 2 and the nullity is $3 - 2 = 1$.
- (v) The rank is 2 and the nullity is $2 - 2 = 0$.

Exercise 3

Find a subset of the vectors that forms a basis for the subspace spanned by the vectors:

- (i) $(1, 0), (-1, 0), (1, 1)$;

Solution. Write the vectors as columns in a matrix:

$$\begin{pmatrix} 1 & -1 & 1 \\ 0 & 0 & 1 \end{pmatrix}$$

The matrix is already in its row echelon form, so we simply collect the rows at the positions of the 1's - the 1st and the 3rd: $\{(1, 0), (1, 1)\}$.

- (ii) $(1, 0, 2), (-1, 1, 3), (0, 1, 5)$;

Solution. Write the vectors as columns in a matrix and bring it to a row echelon form:

$$\begin{pmatrix} 1 & -1 & 0 \\ 0 & 1 & 1 \\ 2 & 3 & 5 \end{pmatrix} \sim \begin{pmatrix} 1 & -1 & 0 \\ 0 & 1 & 1 \\ 0 & 5 & 5 \end{pmatrix} \sim \begin{pmatrix} 1 & -1 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{pmatrix}.$$

We have the leading 1's at the positions 1 and 2, so we take the 1st and the 2nd columns in the original system: $\{(1, 0, 2), (-1, 1, 3)\}$.

- (iii) $(1, 0, 2, -1), (-1, 1, 3, 0), (0, 1, 5, -1)$;

Solution. Write the vectors as columns in a matrix and bring it to a row echelon form:

$$\begin{pmatrix} 1 & -1 & 0 \\ 0 & 1 & 1 \\ 2 & 3 & 5 \\ -1 & 0 & -1 \end{pmatrix} \sim \begin{pmatrix} 1 & -1 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

We have the leading 1's again at the positions 1 and 2, so we take the 1st and the 2nd columns in the original system: $\{(1, 0, 2, -1), (-1, 1, 3, 0)\}$.

- (iv) $(1, 0, 2, -1, 3), (-1, 1, 3, 0, 0), (0, 1, 5, -1, 1)$;

Solution. Write the vectors as columns in a matrix and bring it to a row echelon form:

$$\begin{pmatrix} 1 & -1 & 0 \\ 0 & 1 & 1 \\ 2 & 3 & 5 \\ -1 & 0 & -1 \\ 3 & 0 & 1 \end{pmatrix} \sim \begin{pmatrix} 1 & -1 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

Now the leading 1's are at the positions 1, 2 and 3, so we take all the 3 columns in the original system: $\{(1, 0, 2, -1, 3), (-1, 1, 3, 0, 0), (0, 1, 5, -1, 1)\}$.