Course 2E1 2004-05 (SF Engineers & MSISS & MEMS)

S h e e t 15

Due: in the tutorial sessions next Wednesday/Thursday

Exercise 1

Which of the following sets of vectors are linearly dependent?

(i) $(0,1)$, $(0,2)$;

Solution. We have to look for nontrivial solutions (k_1, k_2) of the vector equation $k_1(0,1) + k_2(0,2) = 0$ or the system

$$\begin{cases} 0k_1 + 0k_2 = 0 \\ k_1 + 2k_2 = 0 \end{cases},$$

which has solution $(k_1, k_2) = (2, -1) \neq (0,0)$, hence the set is dependent.

(ii) $(0,1)$, $(0,2)$, $(1,2)$;

Solution. Now the equation is $k_1(0,1) + k_2(0,2) + k_3(1,2) = 0$, having e.g. the solution $(2,-1,0)$, hence linearly dependent.

(iii) $(0,1,0)$, $(0,2,1)$, $(1,2,0)$;

Solution. Now the equation is $k_1(0,1,0) + k_2(0,2,1) + k_3(1,2,0) = 0$, i.e.

$$\begin{cases} 0k_1 + 0k_2 + k_3 = 0 \\ k_1 + 2k_2 + 2k_3 = 0 \\ 0k_1 + k_2 + 0k_3 = 0, \end{cases}$$

from where we have $k_1 = k_2 = k_3 = 0$, hence the only solution is trivial and the set is independent.

(iv) $(0,1,-1)$, $(1,2,0)$, $(1,0,2)$;

Solution. Here the system becomes

$$\begin{cases} 0k_1 + k_2 + k_3 = 0 \\ k_1 + 2k_2 + 0k_3 = 0 \\ -k_1 + 0k_2 + 2k_3 = 0, \end{cases}$$

which has the nontrivial solution $(k_1, k_2, k_3) = (2, -1, 1)$ and hence the set is dependent.

(v) $(0, 0, 0, 0)$, $(1, 1, 1, 1)$.

Solution. The equation $k_1(0, 0, 0, 0) + k_2(1, 1, 1, 1) = 0$ has clearly the nontrivial solution $(k_1, k_2) = (1, 0)$, so the set is again dependent.

Exercise 2

For which real values of λ do the following vectors form a linearly dependent set in $\mathbb{R}^3$?

$$\mathbf{v}_1 = (\lambda, 1, 1), \quad \mathbf{v}_2 = (1, \lambda, 1), \quad \mathbf{v}_3 = (1, 1, \lambda).$$

Solution. Here λ is a parameter but we still have to look for nontrivial solutions of the vector equation $k_1\mathbf{v}_1 + k_2\mathbf{v}_2 + k_3\mathbf{v}_3 = 0$, i.e. the system

$$\begin{cases} k_1\lambda + k_2 + k_3 = 0 \\ k_1 + k_2\lambda + k_3 = 0 \\ k_1 + k_2 + k_3\lambda = 0. \end{cases}$$

An efficient way to look for solutions is to write the matrix and use the method of the elementary row operations. The matrix is

$$\begin{pmatrix} \lambda & 1 & 1 \\ 1 & \lambda & 1 \\ 1 & 1 & \lambda \end{pmatrix}.$$

Now to get a leading '1' in the upper left corner, we can exchange the 1st and the 3rd rows (we could also divide the first row by λ but this is only possible for $\lambda = 0$, so we would have to consider separate cases):

$$\begin{pmatrix} 1 & 1 & \lambda \\ 1 & \lambda & 1 \\ \lambda & 1 & 1 \end{pmatrix}.$$

Next, subtracting the 1st row from the 2nd, and the 1st times λ from the 3rd, we eliminate both leading entries for the 2nd and the 3rd row:

$$\begin{pmatrix} 1 & 1 & \lambda \\ 0 & \lambda-1 & 1-\lambda \\ 0 & 1-\lambda & 1-\lambda^2 \end{pmatrix}.$$

Now we see that the 2nd row is divisible by $\lambda - 1$ in case $\lambda \neq 1$. The other case $\lambda = 1$ is easy - only the 1st equation becomes nontrivial and has many nontrivial solutions. Hence, for $\lambda = 1$, the set is dependent.

In the next we assume $\lambda \neq 1$ and divide both the 2nd and the 3rd rows by $\lambda - 1$:

$$\begin{pmatrix} 1 & 1 & \lambda \\ 0 & 1 & -1 \\ 0 & -1 & -1-\lambda \end{pmatrix}.$$

Finally we eliminate the leading '-1' in the 3rd row by adding the 2nd row to it:

$$\begin{pmatrix} 1 & 1 & \lambda \\ 0 & 1 & -1 \\ 0 & 0 & -2-\lambda \end{pmatrix}.$$

Now the original system is equivalent to

$$\begin{cases} k_1 + k_2 + k_3\lambda = 0 \\ k_2 - k_3 = 0 \\ -(2+\lambda)k_3 = 0. \end{cases}$$

We now see that the last equation has nontrivial solutions k_3 if the coefficient is zero, i.e. $\lambda = -2$. In that case we can put e.g. $k_3 = 1$ and get k_2 from the 2nd equation, then k_1 from the 1st equation. This means, we find a nontrivial solution and the set is dependent. Otherwise, if $\lambda \neq 2$, the system only has the trivial solution $k_1 = k_2 = k_3 = 0$ and hence the set is independent.

Summarizing, we have that the set is dependent if and only if λ is 1 or -2.

Exercise 3

Which of the following sets of vectors are bases for the corresponding space $\mathbb{R}^n$? (The dimension n should be clear from the length of vectors.)

(i) $(1, 0)$;

Solution. We have 1 vector in $\mathbb{R}^2$, whose dimension is 2, so we would need 2 vectors for a basis and 1 vector set cannot be a basis.

(ii) $(1, 0)$, $(1, 1)$;

Solution. Here the number of vectors is 2 and the dimension is 2, so they form a basis if and only if they are linearly independent. The latter property is checked by solving the system

$$\begin{cases} k_1 + k_2 = 0 \\ k_2 = 0, \end{cases}$$

which only has the trivial solution, meaning that the set is independent and hence a basis.

(iii) $(1, -1)$, $(-1, 1)$;

Solution. Here we have again 2 vectors in the space of dimension 2 but they are linearly dependent (the vector equation $k_1(1, -1) + k_2(-1, 1) = 0$ has nontrivial solutions), hence not a basis.

(iv) $(1, 1)$, $(1, -1)$, $(-1, -1)$;

Solution. Here we have 3 vectors in the space of dimension 2, no way to be a basis.

(v) $(1, 0, 0, 1)$, $(1, 2, 3, 4)$, $(4, 3, 2, 1)$;

Solution. Here again the number of vectors is 3 which does not match the dimension 4 of the space, not a basis.

(vi) $(1, 0, 1)$, $(0, 1, 1)$, $(1, 1, 0)$.

Solution. Here the number of vectors 3 does match the dimension, so we have to check them for linear dependence. The checking is as before and shows that the set is independent and hence a basis.