MATHSOC

TRINITY COLLEGE

GENERAL RELATIVITY:

The weak equivalence principle

Fermín Viniegra

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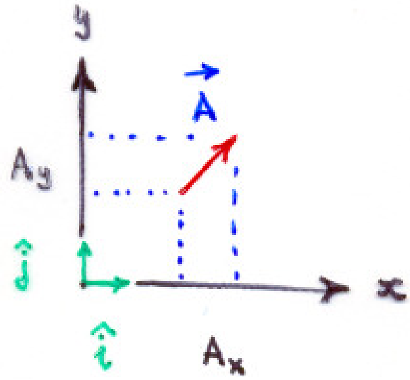
Index Notation

2-D:



3 ingredients $\begin{cases} \vec{A} \\ A_x, A_y \\ \hat{i}, \hat{j} \end{cases}$

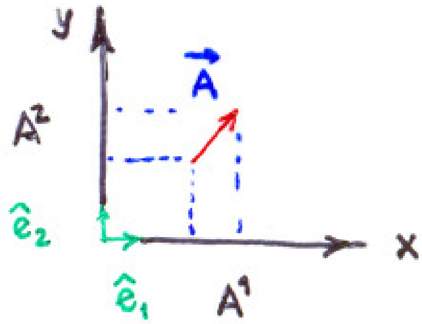
$$\boxed{\vec{A} = A_x \hat{i} + A_y \hat{j}}$$

Just a change of notation

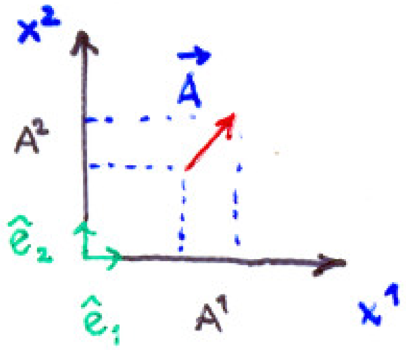
$$= A^x \hat{i} + A^y \hat{j}$$

$$= A^x \hat{e}_x + A^y \hat{e}_y$$

$$\vec{A} = A^1 \hat{e}_1 + A^2 \hat{e}_2$$



Full indexised picture:



Coordinates:

$$x^1, \; x^2$$

vector:

$$\boxed{\vec{A} = \sum_{i=1}^{2} A^i \hat{e}_i}$$

Einstein Index Convention

from $\vec{A} = \sum_{i=1}^{2} A^i \hat{e}_i$ to $\vec{A} = A^i \hat{e}_i$

repited indices indicate sum

Example
3-D

$$K^i{}_j M^j{}_k = K^i{}_1 M^1{}_k + K^i{}_2 M^2{}_k + K^i{}_3 M^3$$

free indices $\quad y^k = A^k{}_i \, x^i \quad$ dummy indices

Rules:

1) No more than 2 indices in each term

$$A^k{}_k x^k \quad ✗$$

2) Free indices must be balanced through out the equation

$$y^k = A^l{}_i \, x^i \quad ✗$$

3) dummy indices can change name :

$$A^k x_k = A^i x_i = A^s x_s \quad ✓$$

The Inner product & the Metric

In 2-D:

$$\vec{A}\cdot\vec{B} = A_x B_x + A_y B_y$$
$$= A^1 B_1 + A^2 B_2$$
$$= A^i B_i \quad \cdots ①$$

$$\vdots$$
$$= A_i B^i$$

On the other hand:

$$\vec{A}\cdot\vec{B} = (A^i \hat{e}_i)\cdot(B^j \hat{e}_j)$$
$$= (\hat{e}_i\cdot\hat{e}_j)(A^i B^j) \quad \cdots ②$$

components are just numbers

Introducing the metric of a 2-D Rectangular Coordinate System:

$$\boxed{g_{ij} \equiv \hat{e}_i \cdot \hat{e}_j}$$ 2-D metric $\cdots ③$

where $\hat{e}_1 \equiv \hat{i}$ and $\hat{e}_2 = \hat{j}$

Properties:

1) $g_{ij} M^j = M_i$ Lowers the index

2) $g^{ik} g_{kj} = \delta^i_j$ inverse

3) $g^{ij} N_j = N^i$ raise the index

Using ③ in ②

$$\vec{A}\cdot\vec{B} = g_{ij} A^i B^j$$
$$= A^i B_j = A_j B^j$$

δ^i_j the kronecker delta. Index form of the identity matrix

Metric & sizes

the "dot" product allow us to calculate:

$$\vec{A} \cdot \vec{A} = \| A \|^2 \qquad \text{the norm of } \vec{A}$$

inside there is the metric allowing this to happen

How to calculate the components of the metric in other coordinate systems?

Ans: use formula $\hat{e}_i \cdot \hat{e}_j$

+ Cartesian

$$\hat{i}\cdot\hat{i} = 1 \quad \hat{i}\cdot\hat{j} = 0$$
$$\hat{j}\cdot\hat{i} = 0 \quad \hat{j}\cdot\hat{j} = 1 \quad \Rightarrow (g_{ij}) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

+ Spherical coordinates 3D

$$\hat{e}_1 = \hat{e}_r \; ; \; \hat{e}_2 = \hat{e}_\theta \; ; \; \hat{e}_3 = \hat{e}_\phi$$

$$\hat{e}_r\cdot\hat{e}_r = 1 \qquad \hat{e}_r\cdot\hat{e}_\theta = 0 \qquad \hat{e}_r\cdot\hat{e}_\phi = 0$$

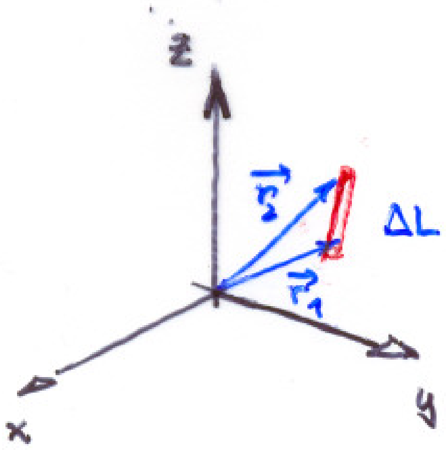
$$\hat{e}_\theta\cdot\hat{e}_r = 0 \qquad \hat{e}_\theta\cdot\hat{e}_\theta = r^2 \qquad \hat{e}_\theta\cdot\hat{e}_\phi = 0$$

$$\hat{e}_\phi\cdot\hat{e}_r = 0 \qquad \hat{e}_\phi\cdot\hat{e}_\theta = 0 \qquad \hat{e}_\phi\cdot\hat{e}_\phi = r^2\sin^2\theta$$

$$\Rightarrow \quad (g_{ij}) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & r^2 & 0 \\ 0 & 0 & r^2\sin^2\theta \end{pmatrix}$$

The length and the metric

Find the length of the rod in the next figure



$\vec{r}_2 - \vec{r}_1$ – relative distance between end points of the rod

ΔL – The length of the rod

The Length is going to be:

$$\Delta L = (\vec{r}_2 - \vec{r}_1) \cdot (\vec{r}_2 - \vec{r}_1) = \Delta\vec{r} \cdot \Delta\vec{r}$$

$$= g_{ij}\, \Delta x^i \Delta x^j$$

So:

$$\Delta L = g_{ij}\, \Delta x^i \Delta x^j$$

for an infinitesimal rod

$$\boxed{dL = g_{ij}\, dx^i dx^j}$$

Point made:

- Lengths are related to the metric
- The metric is related to the coordinate system.

Sequence:

coordinate system $\longrightarrow$ basis vectors

$\longrightarrow$ metric of that space

$\longrightarrow$ norm of vectors

$\longrightarrow$ length between points