Sample Scholarship Exam Questions, Linear Algebra

Disclaimer: this list of questions is intended to reflect actual difficulty of Schol questions rather than actual topics. Refer to the last year's syllabus and "list of sample questions" to recall the topics that may be covered. The actual paper will contain **three** linear algebra questions.

1. (a) Define elementary row operations and elementary matrices. Show that if we multiply an $m \times n$-matrix A by some elementary $m \times m$-matrix B on the left, the result is obtained from A by the elementary row operation corresponding to B.

   (b) Using elementary row operations, compute A^{-1}, where

$$A = \begin{pmatrix} 1 & -5 & 1 \\ -1 & -2 & 3 \\ -6 & 1 & 11 \end{pmatrix}$$

2. Compute the determinant of the $n \times n$-matrix $\begin{pmatrix} 0 & 1 & 1 & \dots & 1 \\ 1 & 0 & 1 & \dots & 1 \\ \vdots & \dots & \ddots & \dots & \vdots \\ \vdots & \dots & \dots & \ddots & \vdots \\ 1 & 1 & \dots & 1 & 0 \end{pmatrix}$ (all diagonal entries are equal to 0, all off-diagonal entries are equal to 1).

3. A $n \times n$-matrix A satisfies the equation $A^2 = A$.

   (a) List all possible characteristic polynomials of A.

   (b) Show that A is similar to a diagonal matrix.

4. Consider the 4-dimensional vector space V consisting of all 2×2-matrices with complex entries (with obvious operations of addition and multiplication by complex numbers). The matrix $A = \begin{pmatrix} 1 & -1 \\ 1 & -1 \end{pmatrix}$ defines a mapping $L\colon V \to V$ by the formula $L(X) = AX - XA$.

   (a) Show that L is a linear operator and describe its eigenvalues and eigenvectors.

   (b) Determine the Jordan normal form of L and find some Jordan basis.

5. Assume that for a $n \times n$-matrix A with real matrix elements we have $A^2 = -I$. Prove that $\operatorname{tr} A = 0$.

6. Prove that if for two square matrices A and B we have $AB - BA = A$, then $\det(A) = 0$.

7. (a) Write down the definition of a bilinear form on a real vector space. Which bilinear forms are said to be positive definite?

   (b) Consider the vector space V of all polynomials in t of degree at most 2. The bilinear form ϕ_a on V (depending on a [real] parameter a) is defined by the formula

$$\phi_a(f(t), g(t)) = \int_{-1}^{1} f(t)g(t)(t^2 - at)\, dt.$$

   Determine all values of a for which ϕ_a is positive definite.