

1. First general observation: each of these congruences has at most two solutions: modulo 13 it is true because in a field a quadratic equation has at most two roots, and modulo $169 = 13^2$ and $2197 = 13^3$, we use Hensel's lemma and the fact that $(x^2 + 3)' = 2x$ has no common roots with $x^2 + 3$ modulo 13.

(a) Clearly, 6 and $-6 \equiv 7 \pmod{13}$ are solutions, and by our observations these are the only solutions.

(b) Let us apply the approach adopted in the proof of Hensel's lemma: we solve the congruence $(6 + 13h)^2 \equiv -3 \pmod{169}$, or, equivalently, $13(3 + 12h) \equiv 0 \pmod{13^2}$, $3 + 12h \equiv 0 \pmod{13}$, $h = 3$, and the solution lifting $x = 6$ is $x = 45$. The other solution is, clearly, $-45 \equiv 124 \pmod{169}$.

(c) Let us apply the approach adopted in the proof of Hensel's lemma: we solve the congruence $(45 + 13^2h)^2 \equiv -3 \pmod{13^3}$, or, equivalently, $13^2(12 + 90h) \equiv 0 \pmod{13^2}$, $12 + 90h \equiv 0 \pmod{13}$, $h = 12$, and the solution lifting $x = 45$ is $x = 2073$. The other solution is, clearly, $-2073 \equiv 124 \pmod{13^3}$.

2. (a) Let us apply the approach adopted in the proof of Hensel's lemma for the equation $10x - 1 = 0$. Modulo 11 we have the solution $x = 10$. Continuing, we have $10(10 + 11h) - 1 \equiv 0 \pmod{121}$, that is $11(9 + 10h) \equiv 0 \pmod{121}$, or $9 + 10h \equiv 0 \pmod{11}$, $h = 9$. Furthermore, we have $10(10 + 11 \cdot 9 + 11^2h) - 1 \equiv 0 \pmod{11^3}$, that is $11^2(9 + 10h) \equiv 0 \pmod{11^3}$, or $9 + 10h \equiv 0 \pmod{11}$, $h = 9$. Continuing like that, we obtain

$$\frac{1}{10} = 10 + 9 \cdot 11 + 9 \cdot 11^2 + 9 \cdot 11^3 + \dots$$

(b) Let us apply the approach adopted in the proof of Hensel's lemma for the equation $16x + 9 = 0$. Modulo 13 we have the solution $x = 10$. Continuing, we have $16(10 + 13h) + 9 \equiv 0 \pmod{169}$, that is $13(13 + 16h) \equiv 0 \pmod{169}$, or $0 + 16h \equiv 0 \pmod{13}$, $h = 0$. Furthermore, we have $16(10 + 13^2h) + 9 \equiv 0 \pmod{13^3}$, that is $13^2(1 + 16h) \equiv 0 \pmod{13^3}$, or $1 + 16h \equiv 0 \pmod{13}$, $h = 4$. On the next step, we have

$$16(10 + 4 \cdot 13^2 + 13^3h) + 9 \equiv 0 \pmod{13^4},$$

that is $13^3(5 + 16h) \equiv 0 \pmod{13^4}$, or $5 + 16h \equiv 0 \pmod{13}$, $h = 7$. On the next step, we have

$$16(10 + 4 \cdot 13^2 + 7 \cdot 13^3 + 13^4h) + 9 \equiv 0 \pmod{13^5},$$

that is $13^4(9 + 16h) \equiv 0 \pmod{13^5}$, or $9 + 16h \equiv 0 \pmod{13}$, $h = 10$, and from here the pattern starts repeating itself, so we get

$$-\frac{9}{16} = 10 + 4 \cdot 13^2 + 7 \cdot 13^3 + 10 \cdot 13^4 + 4 \cdot 13^6 + 7 \cdot 13^7 + \dots$$

3. In this question, instead of mimicking Hensel's lemma, we cheat a little bit and use the properties of the specific numbers given (though it is not too difficult to deal with it in exactly the same way as in the previous question; the most important thing is that after factoring out a power of p from the denominator everything happens inside \mathbb{Z}_p):

(a)

$$\begin{aligned} 1/24 &= 1/8 \cdot 1/3 = 1/8(1 - 2/3) = 1/8(1 + 2/(1 - 4)) = \\ &= 1/8(1 + 2 + 2^3 + 2^5 + 2^7 + \dots) = \\ &= 2^{-3} + 2^{-2} + 1 + 2^2 + 2^4 + 2^6 + \dots \end{aligned}$$

(b)

$$\begin{aligned} 1/120 &= 1/5 \cdot 1/24 = 1/5(24 - \frac{23 \cdot 25}{24}) = 1/5(4 + 4 \cdot 5 + \frac{23 \cdot 25}{1 - 25}) = \\ &= 1/5(4 + 4 \cdot 5 + 23 \cdot 25 + 23 \cdot 25^2 + \dots) = \\ &= 1/5(4 + 4 \cdot 5 + (3 + 4 \cdot 5) \cdot 25 + (3 + 4 \cdot 5) \cdot 25^2 + \dots) = \\ &= \frac{4}{5} + 4 + 3 \cdot 5 + 4 \cdot 25 + 3 \cdot 5^3 + 4 \cdot 5^4 + \dots \end{aligned}$$

4. From Homework 2, we know that $\left(\frac{-3}{p}\right) = \left(\frac{p}{3}\right)$, so for odd $p = 3k+1$ the equation $x^2+3 = 0$ has solutions in \mathbb{Z}_p by Hensel's lemma. For $p \neq 3k+1$, the mapping $x \mapsto x^3$ of the multiplicative group of nonzero remainders modulo p is injective: by Fermat's last theorem, the order of an element divides the order of the group, that is $p - 1$, so 3 cannot be an order of an element of that group. An injective map of a finite set into itself has to be surjective, so every element is a cube. If $p \neq 3$, that is enough to find a solution in \mathbb{Z}_p , because $(x^3 - 37)' = 3x^2$ which has no common roots with $x^3 - 37$. For $p = 3$ the situation is more subtle. Note that for $x = 4$ we have $x^3 - 37 = 27$, so we have a root modulo 3^3 which is not a root of the derivative modulo 3^2 , and the strong version of Hensel's lemma works.

5. Let us first handle case $p > 11$. Let us look at numbers $-1, 2, 3, 5, 11$ modulo p . If -1 is a non-square, then either 15 or -15 is a square, and given that for $y = z = 1$ our equation becomes $x^2 = 15$, and for $y = 2, z = 1$ our equation becomes $x^2 = -15$, we get a solution that can be lifted to \mathbb{Z}_p . Thus,

we may assume that -1 is a square. Also, it follows from our consideration that we may assume that 15 is a non-square. Now, if 3 is a square, then we can find a solution y_0 to the equation $y^2 = 3$, and from our assumption it is clear that we can find a solution x_0 to the equation $x^2 = -1$. The triple $(x_0, y_0, 1)$ solves the original equation, because $-1 + 18 - 17 = 0$. Therefore 3 is a non-square, and $5 = 15 \cdot 3/3^2$ is a square. Also, if 2 is a square, then $-2 = (-1) \cdot 2$ is a square as well, and for $y = 1, z = 0$ we have an equation $x^2 = -2$, so we can solve this equation and lift a solution to \mathbb{Z}_p . If 2 is not a square, and 11 is not a square, we see that $110 = 2 \cdot 5 \cdot 11$ is a square. But for $y = 3, z = 2$ our equation becomes $x^2 = 110$, and we are done. Finally, if 11 is not a square, then $-33 = (-1) \cdot 3 \cdot 11$ is a square, $-33 = x_0^2$. Substituting $x = x_0, y = 1$, we get $2y^4 = 50, y^2 = 5 = 15 \cdot 3/3^2$, which has solutions.

It remains to consider the cases $p = 2, 3, 5, 7, 11$. Clearly, $\left(\frac{15}{7}\right) = \left(\frac{1}{7}\right) = 1$, and

$$\left(\frac{15}{11}\right) = \left(\frac{5}{11}\right) \left(\frac{3}{11}\right) = \left(\frac{11}{5}\right) \cdot (-1) \cdot \left(\frac{11}{3}\right) = -\left(\frac{1}{5}\right) \left(\frac{2}{3}\right) = 1,$$

so the last two cases are covered by setting $y = z = 1$ and lifting x . For $p = 3$ we have a solution $(1, 1, 0)$, and we can lift x . For $p = 2$ we have a solution $(1, 0, 1)$ modulo 16 , and x can be lifted using the strong version Hensel's lemma. Finally, for $p = 5$ we have a solution $(0, 1, 1)$, and lifting y (or z) would work.

6. Idea of the proof: consider the function $f(x)$ which is the sum of the 2-adic series $x + \frac{x^2}{2} + \frac{x^3}{3} + \dots + \frac{x^n}{n} + \dots$. This function is defined for $x \equiv 0 \pmod{2}$ (check that under this condition the partial sums stabilize modulo any power of $2!$). Let us define the function $g(x) := f(1 - x)$. This function, is clearly, defined for $x \equiv 1 \pmod{2}$. Now, it is easy to check that $g(xy) = g(x) + g(y)$. whenever all the values $g(x), g(y), g(xy)$ are defined. (Note that it is not too surprising: in a different context, for $|1 - x| < 1$, where x is a real number, and $|\cdot|$ denotes the usual absolute value, this statement is true because in this case $g(x) = \ln x$.) Now, $g(1) = f(0) = 0$, and $g(1) = g((-1) \cdot (-1)) = g(-1) + g(-1) = 2g(-1)$, so $g(-1) = 0$. This means that the partial sums that converge to $g(0)$ stabilize at zero, that is are divisible by larger and larger powers of 2 .

7. There is no reason to assume that the series for $\sin(p\alpha)$ is convergent to the same value in \mathbb{R} and \mathbb{Q}_p . So the answer is no.