MA 1111/1212: Linear Algebra

Selected answers/solutions to the assignment due March 29

1. The matrix of the corresponding bilinear form is

$$A = \begin{pmatrix} 18+a & 5 & -a-4 \\ 5 & 3 & -2 \\ -a-4 & -2 & a \end{pmatrix}.$$

We have $\Delta_1 = 18 + a$, $\Delta_2 = 3a + 29$, $\Delta_3 = 21a - 40$. All these numbers are positive if and only if

$$a > \frac{40}{21}.$$

2. (**a**) $\Delta_1 = 2$, $\Delta_2 = 3$, $\Delta_3 = 4$. We are looking for a basis of the form

$$\begin{aligned} f_1 &= \alpha_{11}e_1, \\ f_2 &= \alpha_{12}e_1 + \alpha_{22}e_2, \\ f_3 &= \alpha_{13}e_1 + \alpha_{23}e_2 + \alpha_{33}e_3, \end{aligned}$$

imposing equations $A(e_i, f_j) = 0$ for $i < j$, and $A(e_i, f_i) = 1$ for all i. This means that

$$\begin{aligned} 1 &= A(e_1, f_1) = 2\alpha_{11}, \\ 0 &= A(e_1, f_2) = 2\alpha_{12} + \alpha_{22}, \\ 1 &= A(e_2, f_2) = \alpha_{12} + 2\alpha_{22}, \\ 0 &= A(e_1, f_3) = 2\alpha_{13} + \alpha_{23}, \\ 0 &= A(e_2, f_3) = \alpha_{13} + 2\alpha_{23} + \alpha_{33}, \\ 1 &= A(e_3, f_3) = \alpha_{23} + 2\alpha_{33}. \end{aligned}$$

Solving these linear equations, we get $\alpha_{11} = \frac{1}{2}$, $\alpha_{12} = -\frac{1}{3}$, $\alpha_{22} = \frac{2}{3}$, $\alpha_{13} = \frac{1}{4}$, $\alpha_{23} = -\frac{1}{2}$, $\alpha_{33} = \frac{3}{4}$, so the required change of basis is

$$\begin{aligned} f_1 &= \frac{1}{2}e_1, \\ f_2 &= -\frac{1}{3}e_1 + \frac{2}{3}e_2, \\ f_3 &= \frac{1}{4}e_1 - \frac{1}{2}e_2 + \frac{3}{4}e_3. \end{aligned}$$

(**b**) $\Delta_1 = 1$, $\Delta_2 = -2$, $\Delta_3 = 3$. We are looking for a basis of the form

$$\begin{aligned} f_1 &= \alpha_{11}e_1, \\ f_2 &= \alpha_{12}e_1 + \alpha_{22}e_2, \\ f_3 &= \alpha_{13}e_1 + \alpha_{23}e_2 + \alpha_{33}e_3, \end{aligned}$$

imposing equations $A(e_i, f_j) = 0$ for $i < j$, and $A(e_i, f_i) = 1$ for all i. This means that

$$\begin{aligned} 1 &= A(e_1, f_1) = \alpha_{11}, \\ 0 &= A(e_1, f_2) = \alpha_{12} + 2\alpha_{22}, \\ 1 &= A(e_2, f_2) = 2\alpha_{12} + 2\alpha_{22}, \\ 0 &= A(e_1, f_3) = \alpha_{13} + 2\alpha_{23}3\alpha_{33}, \\ 0 &= A(e_2, f_3) = 2\alpha_{13} + 2\alpha_{23} + 3\alpha_{33}, \\ 1 &= A(e_3, f_3) = 3\alpha_{13} + 3\alpha_{23} + 3\alpha_{33}. \end{aligned}$$

Solving these linear equations, we get $\alpha_{11} = 1$, $\alpha_{12} = 1$, $\alpha_{22} = -\frac{1}{2}$, $\alpha_{13} = 0$, $\alpha_{23} = 1$, $\alpha_{33} = -\frac{2}{3}$, so the required change of basis is

$$\begin{aligned} f_1 &= e_1, \\ f_2 &= e_1 - \frac{1}{2}e_2, \\ f_3 &= e_2 - \frac{2}{3}e_3. \end{aligned}$$

(**c**) $\Delta_1 = 1$, $\Delta_2 = 1$, $\Delta_3 = 1$. We are looking for a basis of the form

$$\begin{aligned} f_1 &= \alpha_{11}e_1, \\ f_2 &= \alpha_{12}e_1 + \alpha_{22}e_2, \\ f_3 &= \alpha_{13}e_1 + \alpha_{23}e_2 + \alpha_{33}e_3, \end{aligned}$$

imposing equations $A(e_i, f_j) = 0$ for $i < j$, and $A(e_i, f_i) = 1$ for all i. This means that

$$\begin{aligned} 1 = A(e_1, f_1) &= \alpha_{11}, \\ 0 = A(e_1, f_2) &= \alpha_{12} + \alpha_{22}, \\ 1 = A(e_2, f_2) &= \alpha_{12} + 2\alpha_{22}, \\ 0 = A(e_1, f_3) &= \alpha_{13} + \alpha_{23} + \alpha_{33}, \\ 0 = A(e_2, f_3) &= \alpha_{13} + 2\alpha_{23} + 2\alpha_{33}, \\ 1 = A(e_3, f_3) &= \alpha_{13} + 2\alpha_{23} + 3\alpha_{33}. \end{aligned}$$

Solving these linear equations, we get $\alpha_{11} = 1$, $\alpha_{12} = -1$, $\alpha_{22} = 1$, $\alpha_{13} = 0$, $\alpha_{23} = -1$, $\alpha_{33} = 1$, so the required change of basis is

$$\begin{aligned} f_1 &= e_1, \\ f_2 &= -e_1 + e_2, \\ f_3 &= -e_2 + e_3. \end{aligned}$$

(**d**) $\Delta_1 = 3$, $\Delta_2 = 8$, $\Delta_3 = 20$, $\Delta_4 = 48$. We are looking for a basis of the form

$$\begin{aligned} f_1 &= \alpha_{11}e_1, \\ f_2 &= \alpha_{12}e_1 + \alpha_{22}e_2, \\ f_3 &= \alpha_{13}e_1 + \alpha_{23}e_2 + \alpha_{33}e_3, f_4 &= \alpha_{14}e_1 + \alpha_{24}e_2 + \alpha_{34}e_3 + \alpha_{44}e_4, \end{aligned}$$

imposing equations $A(e_i, f_j) = 0$ for $i < j$, and $A(e_i, f_i) = 1$ for all i. This means that

$$\begin{aligned} 1 = A(e_1, f_1) &= 3\alpha_{11}, \\ 0 = A(e_1, f_2) &= 3\alpha_{12} + \alpha_{22}, \\ 1 = A(e_2, f_2) &= \alpha_{12} + 3\alpha_{22}, \\ 0 = A(e_1, f_3) &= 3\alpha_{13} + \alpha_{23} + \alpha_{33}, \\ 0 = A(e_2, f_3) &= \alpha_{13} + 3\alpha_{23} + \alpha_{33}, \\ 1 = A(e_3, f_3) &= \alpha_{13} + \alpha_{23} + 3\alpha_{33}, \\ 0 = A(e_1, f_4) &= 3\alpha_{14} + \alpha_{24} + \alpha_{34} + \alpha_{44}, \\ 0 = A(e_2, f_4) &= \alpha_{14} + 3\alpha_{24} + \alpha_{34} + \alpha_{44}, \\ 0 = A(e_3, f_4) &= \alpha_{14} + \alpha_{24} + 3\alpha_{34} + \alpha_{44}, \\ 1 = A(e_4, f_4) &= \alpha_{14} + \alpha_{24} + \alpha_{34} + 3\alpha_{44}. \end{aligned}$$

Solving these linear equations, we get $\alpha_{11} = \frac{1}{3}$, $\alpha_{12} = -\frac{1}{8}$, $\alpha_{22} = \frac{3}{8}$, $\alpha_{13} = -\frac{1}{10}$, $\alpha_{23} = -\frac{1}{10}$, $\alpha_{33} = \frac{2}{5}$, $\alpha_{14} = -\frac{1}{12}$, $\alpha_{24} = -\frac{1}{12}$, $\alpha_{34} = -\frac{1}{12}$, $\alpha_{44} = \frac{5}{12}$, so the required change of basis is

$$\begin{aligned} f_1 &= \frac{1}{3}e_1, \\ f_2 &= -\frac{1}{8}e_1 + \frac{3}{8}e_2, \\ f_3 &= -\frac{1}{10}e_1 - \frac{1}{10}e_2 + \frac{2}{5}e_3, \\ f_4 &= -\frac{1}{12}e_1 - \frac{1}{12}e_2 - \frac{1}{12}e_3 + \frac{5}{12}e_4. \end{aligned}$$

3. If f is skew-symmetric, we have $f(v,v) = -f(v,v)$, and hence $f(v,v) = 0$. If $f(v,v) = 0$ for all v, we have $0 = f(v+w,v+w) = f(v,v)+f(v,w)+f(w,v)+f(w,w)$ and hence $f(v,w)+f(w,v) = 0$.

4. First of all, for the last term $\sin^3(x_1 + x_2 + x_3)$ all second derivatives at the origin are clearly equal to zero (this expression is the cube of something that vanishes at the origin; differentiating reduces the order of zero by one, so the second derivatives of the cube are still equal to zero). Thus, from now on we replace f by $g = \sin(x_1 - x_2)\sin(x_1 - x_3) + \sin^2 x_2 + c\sin x_3 \sin(2x_3)$. We have

$\frac{\partial g}{\partial x_1} = \sin(x_1-x_2)\cos(x_1-x_3)+\cos(x_1-x_2)\sin(x_1-x_3) = \sin(x_1-x_2+x_1-x_3) = \sin(2x_1-x_2-x_3)$,

$\frac{\partial g}{\partial x_2} = -\cos(x_1 - x_2)\sin(x_1 - x_3) + 2\sin(x_2)\cos(x_2) = -\cos(x_1 - x_2)\sin(x_1 - x_3) + \sin(2x_2)$,

$\frac{\partial g}{\partial x_3} = -\sin(x_1 - x_2)\cos(x_1 - x_3) + c(\cos(x_3)\sin(2x_3) + 2\sin(x_3)\cos(2x_3))$,

$\frac{\partial^2 g}{\partial x_1^2} = 2\cos(2x_1 - x_2 - x_3)$,

$\frac{\partial^2 g}{\partial x_1 \partial x_2} = -\cos(2x_1 - x_2 - x_3)$,

$\frac{\partial^2 g}{\partial x_1 \partial x_3} = -\cos(2x_1 - x_2 - x_3)$,

$\frac{\partial^2 g}{\partial x_2 \partial x_1} = -\cos(2x_1 - x_2 - x_3)$,

$\frac{\partial^2 g}{\partial x_2^2} = -\sin(x_1 - x_2)\sin(x_1 - x_3) + 2\cos(2x_2)$,

$\frac{\partial^2 g}{\partial x_2 \partial x_3} = \cos(x_1 - x_2)\cos(x_1 - x_3)$,

$\frac{\partial^2 g}{\partial x_3 \partial x_1} = -\cos(2x_1 - x_2 - x_3)$,

$\frac{\partial^2 g}{\partial x_3 \partial x_2} = \cos(x_1 - x_2)\cos(x_1 - x_3)$,

$$\begin{aligned} \frac{\partial^2 g}{\partial x_3^2} = &-\sin(x_1 - x_2)\sin(x_1 - x_3) + c(2\cos(2x_3)\cos(x_3) - \\ &- \sin(x_3)\sin(2x_3) + 2\cos(2x_3)\cos(x_3) - 4\sin(x_3)\sin(2x_3)), \end{aligned}$$

so the answer to (a) is the matrix

$$A = \begin{pmatrix} 2 & -1 & -1 \\ -1 & 2 & 1 \\ -1 & 1 & 4c \end{pmatrix}.$$

(b) Top left corner determinants of this matrix are 2, 3, and $12c - 2$, so by Sylvester's criterion the corresponding quadratic form is positive definite for $c > \frac{1}{6}$.

5. Since all first derivatives of f and of g vanish at the origin, we have

$$f(x + h) = f(x) + \frac{1}{2}A(h,h) + \varepsilon(h),$$

where $\lim\limits_{h\to 0} \frac{\varepsilon(h)}{|h|^2} = 0$. For $c = 1/5 > 1/6$ our form is positive definite, so f has a local minimum at the origin.