SOLUTION TO QUESTION 5 OF THE FINAL EXAM PAPER

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The goal of this short note is to prove the following inequality for any three matrices A, B, and C for which the product ABC is defined:

$$\operatorname{rk}(AB) + \operatorname{rk}(BC) \le \operatorname{rk}(ABC) + \operatorname{rk}(B).$$

We interpret our matrices as linear operators; thus, C is a linear operator from a vector space V_0 to another vector space V_1, B is a linear operator from V_1 to another vector space V_2, and A is a linear operator from V_2 to another vector space V_3. This way this operators are composable, which corresponds to the fact that the product ABC is defined.

Throughout this note, we keep the notation $F|_U$ for the restriction of an operator: if $F\colon V \to W$ is a linear operator and $U \subset V$ is a subspace, $F|_U$ denotes the same linear operator but restricted to U.

Let us note that

$$\operatorname{rk}(AB) = \dim\{AB(x)\colon x \in V_1\} = \\ = \dim\{A(B(x))\colon x \in V_1\} = \dim \operatorname{Im}(A|_{\operatorname{Im} B}) = \operatorname{rk}(A|_{\operatorname{Im} B})$$

and

$$\operatorname{rk}(ABC) = \dim\{ABC(x)\colon x \in V_1\} = \\ = \dim\{A(BC(x))\colon x \in V_1\} = \dim \operatorname{Im}(A|_{\operatorname{Im}(BC)}) = \operatorname{rk}(A|_{\operatorname{Im}(BC)}),$$

so by the formula

$$\operatorname{rk} F + \dim \operatorname{Ker} F = \dim V$$

(well known from the lectures), we get

$$\operatorname{rk}(AB) = \dim(\operatorname{Im} B) - \dim \operatorname{Ker}(A|_{\operatorname{Im} B}) = \operatorname{rk}(B) - \dim \operatorname{Ker}(A|_{\operatorname{Im} B})$$

and

$$\operatorname{rk}(ABC) = \dim(\operatorname{Im}(BC)) - \dim \operatorname{Ker}(A|_{\operatorname{Im}(BC)}) = \operatorname{rk}(BC) - \dim \operatorname{Ker}(A|_{\operatorname{Im}(BC)}).$$

Substituting these into the inequality we want to prove, we see that it is equivalent to the inequality

$$\dim \operatorname{Ker}(A|_{\operatorname{Im}(BC)}) \le \dim \operatorname{Ker}(A|_{\operatorname{Im} B})$$

which is obvious: $\operatorname{Im}(BC)$ clearly is a subset of $\operatorname{Im} B$, therefore $\operatorname{Ker}(A|_{\operatorname{Im}(BC)}) \subset \operatorname{Ker}(A|_{\operatorname{Im} B})$. □

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