# UNIVERSITY OF DUBLIN
## TRINITY COLLEGE

Faculty of Science

school of mathematics

**JF Mathematics** **Trinity Term 2010**
**JF Theoretical Physics**
**JF Two Subject Mod**

Course 1111/1212, a sample exam paper

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For each task, the number of points you can get for a complete solution of that task is printed next to it.

You may use all statements proved in class and in home assignments; when using some statement, you should formulate it clearly, e.g. "in class, we proved that if $A$ is invertible, then the reduced row echelon form of $A$ is the identity matrix".

Maths and TP students should attempt all questions. TSM students should attempt all questions from Part A.

Marks for Part A count towards the mark for the 1111 course, marks for part B count towards the mark for the 1212 course.

All vector spaces unless otherwise specified are over complex numbers.

Non-programmable calculators are permitted for this examination.

PART A

1. Denote by $A$ the matrix $\begin{pmatrix} 1 & 1 & 1 \\ 1 & 2 & 2 \\ 1 & 2 & 3 \end{pmatrix}$ and by $b$ the vector $\begin{pmatrix} 5 \\ -1 \\ 2 \end{pmatrix}$.

   (a) (5 points) List all minors and all cofactors of $A$, and write down the expansion of $\det(A)$ along the second row and along the third column.

   (b) (5 points) Show how to use the Cramer's rule to solve the system $Ax = b$.

2. (10 points) Describe all possible values of $i$, $j$, $k$ and $l$ for which the term

$$a_{4k}a_{35}a_{il}a_{67}a_{j1}a_{23}a_{14}$$

   occurs in the expansion of a $7 \times 7$ determinant with coefficient $-1$.

3. (a) (10 points) Under what conditions is a system of vectors of a vector space $V$ called a basis of this space? Show that if we extend a basis of a vector space by any vector of that space, the resulting system of vectors is not a basis anymore.

   (b) (8 points) Show that if a system of $100$ vectors in $\mathbb{R}^{100}$ is linearly independent, then it is a basis of $\mathbb{R}^{100}$. Show that if a system of $100$ vectors in $\mathbb{R}^{100}$ is complete, then it is a basis of $\mathbb{R}^{100}$.

4. (a) (12 points) Consider the vector space $V$ of all $2\times 2$-matrices (with obvious addition and multiplication by scalars). Show that for every $2 \times 2$-matrix $A$ the mapping $L_A \colon V \to V$ given by the formula $L_A(X) = AX - XA$, is a linear operator. In the case $A = \begin{pmatrix} 2 & -1 \\ 1 & 2 \end{pmatrix}$, write down the matrix of $L_A$ relative to the basis $E_{11} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$, $E_{12} = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$, $E_{21} = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$, $E_{22} = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$.

PART B

5. (12 points) Let $V$ be a vector space. Show that for every two linear operators $A\colon V \to V$ and $B\colon V \to V$ we have

$$\mathrm{rk}(AB) \leq \mathrm{rk}(A) \quad \text{and} \quad \mathrm{rk}(AB) \leq \mathrm{rk}(B).$$

Show that if $B$ is invertible, then $\mathrm{rk}(BA) = \mathrm{rk}(A)$, and give an example showing that this equality might hold even if $B$ is not invertible.

6. (a) (12 points) Determine the Jordan normal form and find some Jordan basis for the matrix

$$A = \begin{pmatrix} 9 & 5 & 2 \\ -16 & -9 & -4 \\ 2 & 1 & 1 \end{pmatrix}$$

(b) (12 points) Find a closed formula for $A^n$.

7. (a) (2 points) Write down the definition of a bilinear form on a real vector space. Which symmetric bilinear forms are said to be positive definite?

(b) (12 points) Define a quadratic form $Q$ on the space $\mathbb{R}^3$ by the formula

$$Q(xe_1 + ye_2 + ze_3) = (20 + 4a)x^2 + 12(1 + a)xz + 6y^2 + 3z^2.$$

Find all values of the parameter $a$ for which this form is positive definite.

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