# Solutions to the midterm questions

1. (a) The reduced row echelon form of the augmented matrix is

$$\begin{pmatrix} 1 & 0 & 0 & 6 \\ 0 & 1 & 0 & -9 \\ 0 & 0 & 1 & 4 \end{pmatrix},$$

so the only solution is $x_1 = 6$, $x_2 = -9$, $x_3 = 4$.

(b) We have

$$A_1 = \begin{pmatrix} 1 & 1 & 1 \\ 0 & 2 & 3 \\ -7 & 5 & 8 \end{pmatrix}, A_2 = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 0 & 3 \\ 1 & -7 & 8 \end{pmatrix}, A_3 = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 2 & 0 \\ 1 & 5 & -7 \end{pmatrix},$$

so $\det(A_1) = -6$, $\det(A_2) = 9$, $\det(A_3) = -4$. Also, $\det(A) = -1$, hence by the Cramer's rule we get the same solution as in (a).

2. (a) Expanding $A_n$ along the first row, and then expanding the second cofactor along the first column, we get precisely the recurrent relation $\det(A_n) = 5\det(A_{n-1}) - 4\det(A_{n-2})$. Note that if we put $\det(A_0) = 1$, this formula is valid for $n = 2$ as well.

(b) For brevity, denote $a_n = \det(A_n)$, $v_n = \begin{pmatrix} a_n \\ a_{n+1} \end{pmatrix}$. Then $v_{n+1} = Av_n$, where $A = \begin{pmatrix} 0 & 1 \\ -4 & 5 \end{pmatrix}$, so $v_n = A^n v_0 = A^n \begin{pmatrix} 1 \\ 5 \end{pmatrix}$. The eigenvalues of $A$ are 1 and 4, with the corresponding eigenvectors $\begin{pmatrix} 1 \\ 1 \end{pmatrix}$ and $\begin{pmatrix} 1 \\ 4 \end{pmatrix}$. Thus, if $C = \begin{pmatrix} 1 & 1 \\ 1 & 4 \end{pmatrix}$, we have $C^{-1}AC = \begin{pmatrix} 1 & 0 \\ 0 & 4 \end{pmatrix}$, and

$$A^n = C \begin{pmatrix} 1 & 0 \\ 0 & 4^n \end{pmatrix} C^{-1} = \begin{pmatrix} \frac{4-4^n}{3} & \frac{4^n-1}{3} \\ \frac{4-4^{n+1}}{3} & \frac{4^{n+1}-1}{3} \end{pmatrix}$$

. Finally, $v_n = A^n v_0 = \begin{pmatrix} \frac{4^{n+1}-1}{3} \\ \frac{4^{n+2}-1}{3} \end{pmatrix}$, and $a_n = \frac{4^{n+1}-1}{3}$.

3. It is easy to see that the characteristic polynomial of this matrix $A$ is $1 - 3t + 3t^2 - t^3 = (1-t)^3$, so all eigenvalues are equal to 1. Also, $A - I = \begin{pmatrix} 2 & -3 & 1 \\ 2 & -3 & 1 \\ 2 & -3 & 1 \end{pmatrix}$ which is a matrix of rank 1, and $(A-I)^2 = 0$. This

means that we have a sequence of spaces $\text{Ker}(A-I) \subset \text{Ker}(A-I)^2$ where the first one is two-dimensional and the second one is three-dimensional. For a relative basis we can take the vector $e = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$. Then $(A-I)e = \begin{pmatrix} 2 \\ 2 \\ 2 \end{pmatrix}$, and it remains to find a vector $f$ which, together with $(A-I)e$, forms a basis of $\text{Ker}(A-I)$. For such a vector we may take $\begin{pmatrix} 3 \\ 2 \\ 0 \end{pmatrix}$. Relative to the basis $f, (A-I)e, e$ our operator has the matrix $\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}$ which is the Jordan normal form we are looking for.

4. (a) We have $\text{tr}(B) = \text{tr}(AB - BA) = \text{tr}(AB) - \text{tr}(BA) = 0$.

(b) The most straightforward way to check that the trace of a matrix is zero is to represent it in the form $AB - BA$. For $B^2$, let us try to compute $AB^2 - B^2A$. We have

$$AB^2 - B^2A = (AB - BA + BA)B - B^2A = \\ = (B + BA)B - B^2A = B^2 + B(AB - BA) = 2B^2,$$

so $2B^2 = AB^2 - B^2A$, $\text{tr}(2B^2) = 0$ and consequently $\text{tr}(B^2) = 0$.

(c) Similarly, we can prove by induction that $AB^k - B^kA = kB^k$, so $\text{tr}(kB^k) = 0$ and $\text{tr}(B^k) = 0$.