Course 1111/1212, sample midterm exam paper

Usual exam conventions apply:

- For each task, the number of points you can get for a complete solution of that task is printed next to it.
- All vector spaces unless otherwise specified are over complex numbers.
- You may use all statements proved in class and in home assignments; when using some statement, you should formulate it clearly, e.g. "in class, we proved that if A is invertible, then the reduced row echelon form of A is the identity matrix".
- *Non-programmable* calculators are permitted.

1. Compute the inverse of the matrix

$$A = \begin{pmatrix} 1 & -5 & 1 \\ -1 & -2 & 3 \\ -2 & 1 & 3 \end{pmatrix}$$

using

   (a) (10 points) elementary row operations;

   (b) (10 points) cofactors and the adjoint matrix;

2. (15 points) Describe all possible values of i, j, k and l for which the term

$$a_{4k}a_{35}a_{il}a_{67}a_{j1}a_{23}a_{14}$$

occurs in the expansion of a 7×7 determinant with coefficient -1.

3. Consider the matrices

$$A = \begin{pmatrix} 9 & 5 & 2 \\ -16 & -9 & -4 \\ 2 & 1 & 1 \end{pmatrix} \quad \text{and} \quad B = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

   (a) (20 points) Describe the Jordan normal form of A and find a Jordan basis for A.

   (b) (10 points) Is A similar to B? Explain your answer.

   (c) (15 points) Find a closed formula for A^n.

4. (20 points) Let V and W be two vector spaces. Show that for every two linear operators A and B from V to W we have

$$\text{rk}(A + B) \leq \text{rk}(A) + \text{rk}(B).$$

Furthermore, show that if $\text{rk}(A + B) = \text{rk}(A) + \text{rk}(B)$, and for some vectors $v_1, v_2 \in V$, $w \in W$ we have $A(v_1) = w = B(v_2)$, then $w = 0$.