UNIVERSITY OF DUBLIN
TRINITY COLLEGE

Faculty of Science

school of mathematics

JF Mathematics **Hilary Term 2010 (midterm)**
JF Theoretical Physics
JF Two Subject Mod

Course 1111/1212

Monday, March 8 SNIAM Lecture Theatre 16.30 – 18.00

Dr. Vladimir Dotsenko

For each task, the number of points you can get for a complete solution of that task is printed next to it.

All vector spaces unless otherwise specified are over complex numbers.

You may use all statements proved in class and in home assignments; when using some statement, you should formulate it clearly, e.g. "in class, we proved that if A is invertible, then the reduced row echelon form of A is the identity matrix".

Non-programmable calculators are permitted for this examination.

1. Solve the system of equations

$$\begin{pmatrix} 1 & 1 & 1 \\ 1 & 2 & 3 \\ 1 & 5 & 8 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ -7 \end{pmatrix}$$

   (a) (12 points) by bringing its augmented matrix to its reduced row echelon form;

   (b) (12 points) by using the Cramer's rule.

2. Let

$$A_n = \begin{pmatrix} 5 & 2 & 0 & 0 & \dots & 0 \\ 2 & 5 & 2 & 0 & \dots & 0 \\ 0 & 2 & 5 & 2 & \dots & 0 \\ \vdots & \vdots & \dots & \ddots & \dots & \vdots \\ 0 & 0 & \dots & 2 & 5 & 2 \\ 0 & 0 & \dots & \dots & 2 & 5 \end{pmatrix}$$

   be the $n \times n$-matrix for which all diagonal elements are equal to 5, all elements on the diagonals next to the main are equal to 2, all other elements are equal to 0.

   (a) (10 points) Show that $\det(A_n) = 5\det(A_{n-1}) - 4\det(A_{n-2})$ for all $n \geq 3$.

   (b) (16 points) Use the recursive formula you obtained to find a closed formula for $\det(A_n)$.

3. (25 points) Determine the Jordan normal form and find some Jordan basis for the matrix

$$\begin{pmatrix} 3 & -3 & 1 \\ 2 & -2 & 1 \\ 2 & -3 & 2 \end{pmatrix}.$$

4. For two $n \times n$-matrices A and B, we have $AB - BA = B$. Show that

   (a) (5 points) $\operatorname{tr}(B) = 0$;

   (b) (10 points) $\operatorname{tr}(B^2) = 0$;

   (c) (10 points) $\operatorname{tr}(B^k) = 0$ for all positive integers k.