Solutions to questions from the fourth tutorial, November 23

1. (**a**) *First solution*: adding $(-x)$ to both sides, we get $(-x)+(x+y) = (-x)+x$, so $((-x)+x)+y = (-x)+x$, $0+y=0$, $y=0$.

Second solution: this equality should hold for all x, in particular $x=0$, and we get $0+y=0$, $y=0$.

(**b**) Assume that a linear combination of the vectors $\mathbf{u}_1$, $\mathbf{v}_1$, $\mathbf{w}_1$ is equal to zero:

$$a\mathbf{u}_1 + b\mathbf{v}_1 + c\mathbf{w}_1 = 0.$$

substituting the expressions for these vectors, we rewrite it as

$$a(\mathbf{v}+\mathbf{w}) + b(\mathbf{u}+\mathbf{w}) + c(\mathbf{u}+\mathbf{v}) = 0,$$

which in turn can be rewritten as

$$(b+c)\mathbf{u} + (a+c)\mathbf{v} + (a+b)\mathbf{w} = 0.$$

Since $\mathbf{u}$, $\mathbf{v}$, and $\mathbf{w}$ are linearly independent, we have to have

$$b+c = a+c = a+b = 0,$$

from which it is easily follows that $a=b=c=0$. (For example, $2a = (a+c)+(a+b)-(b+c) = 0$.) Therefore the only linear combination of our vectors equal to zero is the trivial one, and they are linearly independent.

2. (**a**) (1) is true (0 is an integer), (2) is true (if $\begin{pmatrix} x \\ y \\ z \end{pmatrix}$ has integer coordinates, then $\begin{pmatrix} -x \\ -y \\ -z \end{pmatrix}$ has integer coordinates), (3) is true (if $\begin{pmatrix} x_1 \\ y_1 \\ z_1 \end{pmatrix}$ and $\begin{pmatrix} x_2 \\ y_2 \\ z_2 \end{pmatrix}$ have integer coordinates, then $\begin{pmatrix} x_1+x_2 \\ y_1+y_2 \\ z_1+z_2 \end{pmatrix}$ has integer coordinates), (4) is not true (for example, $\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$ is in U, but if we multiply it by $1/2$, we get $\begin{pmatrix} 1/2 \\ 0 \\ 0 \end{pmatrix}$ which is not in U).

(**b**) (1) is true ($0 \geqslant 0$), (2) is false (for example, $\begin{pmatrix} 1 \\ 0 \end{pmatrix}$ is in U but its opposite $\begin{pmatrix} -1 \\ 0 \end{pmatrix}$ is not), (3) is true (if $\begin{pmatrix} x_1 \\ y_1 \end{pmatrix}$ and $\begin{pmatrix} x_2 \\ y_2 \end{pmatrix}$ have nonnegative coordinates, then $\begin{pmatrix} x_1+x_2 \\ y_1+y_2 \end{pmatrix}$ has nonnegative coordinates), (4) is not true (because we can multiply by negative scalars too, for example, $\begin{pmatrix} 1 \\ 0 \end{pmatrix}$ is in U but $(-1)\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} -1 \\ 0 \end{pmatrix}$ is not).

(**c**) (1) is true ($0^2 - 0\cdot 0 + 0^2 = 0^2$), (2) is true (if $\begin{pmatrix} x \\ y \\ z \end{pmatrix}$ is in U, then

$$(-x)^2 - (-x)(-y) + (-y)^2 = x^2 - xy + y^2 = z^2 = (-z)^2$$

, so the opposite of every vector from U is in U), (3) is not true (for example, $\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$ is in U and $\begin{pmatrix} -1 \\ -1 \\ 1 \end{pmatrix}$ is in U, but their sum $\begin{pmatrix} 0 \\ 0 \\ 2 \end{pmatrix}$ is not in U), (4) is true (if $\begin{pmatrix} x \\ y \\ z \end{pmatrix}$ is in U, then $(cx)^2 - (cx)(cy) + (cy)^2 = c^2(x^2 - xy + y^2) = c^2z^2 = (cz)^2$).

(**d**) (1) is true ($0 = 2 \cdot 0^3$), (2) is true (if $\begin{pmatrix} x \\ y \end{pmatrix}$ is in U, then $(-y) = -2x^3 = 2(-x)^3$), (3) is not true (for example, $\begin{pmatrix} 1 \\ 2 \end{pmatrix}$ is in U, but $\begin{pmatrix} 1 \\ 2 \end{pmatrix} + \begin{pmatrix} 1 \\ 2 \end{pmatrix} = \begin{pmatrix} 2 \\ 4 \end{pmatrix}$ is not in U), (4) is not true (for example, $\begin{pmatrix} 1 \\ 2 \end{pmatrix}$ is in U, but $2\begin{pmatrix} 1 \\ 2 \end{pmatrix} = \begin{pmatrix} 2 \\ 4 \end{pmatrix}$ is not in U).

(**e**) (1) is not true ($0^3 \cdot 0 \neq 1$), (2) is true (if $\begin{pmatrix} x \\ y \end{pmatrix}$ is in U, then $(-x)^3(-y) = x^3y = 1$), (3) is not true (for example, $\begin{pmatrix} 1 \\ 1 \end{pmatrix}$ and $\begin{pmatrix} -1 \\ -1 \end{pmatrix}$ are in U, but $\begin{pmatrix} 1 \\ 1 \end{pmatrix} + \begin{pmatrix} -1 \\ -1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$ is not in U), (4) (3) is not true (for example, $\begin{pmatrix} 1 \\ 1 \end{pmatrix}$ is in U, but $0\begin{pmatrix} 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$ is not in U).

3. We have $c_1e_1 + c_2e_2 = \begin{pmatrix} c_1 + 2c_2 \\ c_1 + c_2 \end{pmatrix}$. If this combination is equal to zero, we have $c_1 + 2c_2 = 0$ and $c_1 + c_2 = 0$, so $c_2 = (c_1 + 2c_2) - (c_1 + c_2) = 0$, and $c_1 = (c_1 + c_2) - c_2 = 0$, so the linear combination has to be trivial and the vectors are linearly independent. Also, for every vector $w = \begin{pmatrix} x \\ y \end{pmatrix}$ we can find a linear combination representing this vector: solving $c_1 + 2c_2 = x$, $c_1 + c_2 = y$, we get $c_2 = x - y$, $c_1 = 2y - x$. Thus this system of vectors is complete. A linearly independent and complete system is a basis. To find coordinates of $\mathbf{v} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ relative to that basis, we have to represent $\mathbf{v}$ as a linear combination of basis vectors. We have $\mathbf{v} = e_2 - e_1 = (-1)e_1 + 1 \cdot e_2$, so the first coordinate is -1 and the second coordinate is 1.