MA 1111/1212: Linear Algebra
Selected answers/solutions to the assignment due December 7, 2010

1. The reduced row echelon of this matrix is $\begin{pmatrix} 1 & 0 & -1 & 1/3 \\ 0 & 1 & 0 & -1/3 \\ 0 & 0 & 0 & 0 \end{pmatrix}$, so x_3 and x_4 are free variables, the dimension of the solution space is 2, the general solution is $\begin{pmatrix} x_3 - \frac{1}{3}x_4 \\ \frac{1}{3}x_4 \\ x_3 \\ x_4 \end{pmatrix} = x_3 \begin{pmatrix} 1 \\ 0 \\ 1 \\ 0 \end{pmatrix} + x_4 \begin{pmatrix} -\frac{1}{3} \\ \frac{1}{3} \\ 0 \\ 1 \end{pmatrix}$, and the latter two vectors form a basis.

2. Denote the entries in the first row by x_1, x_2, x_3, the entries in the second row by x_4, x_5, x_6, and the entries in the third row by x_6, x_7, x_8. We have

$$\begin{aligned} x_1 + x_2 + x_3 &= x_4 + x_5 + x_6, \\ x_4 + x_5 + x_6 &= x_7 + x_8 + x_9, \\ x_1 + x_4 + x_7 &= x_2 + x_5 + x_8, \\ x_1 + x_4 + x_7 &= x_3 + x_6 + x_9, \\ x_1 + x_2 + x_3 &= x_3 + x_6 + x_9 \end{aligned}$$

which gives us five equations involving 9 unknowns. One can compute the reduced row echelon form directly, but let us show how to reduce computations a little bit. Clearly, the last equation is redundant since the sum of all rows is equal to the sum of all columns, so we can keep the first four equations. Since there are four equations, the number of leading variables is at most 4, the number of free variables is at least 5, and the dimension is at least 5. To show that the dimension is equal to 5, it is enough to show that 5 coordinates are enough. Let us assign arbitrary parameters to x_5, x_6, x_7, x_8, and x_9, then from the last row we know what the row/column sum is equal to, from the last column we recover x_3, from the second row we recover x_4, and from the first two columns we recover x_1 and x_2. This leads to the following basis vectors: $\begin{pmatrix} 1 & -1 & 0 \\ -1 & \mathbf{1} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} \end{pmatrix}$, $\begin{pmatrix} 1 & 0 & -1 \\ -1 & \mathbf{0} & \mathbf{1} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} \end{pmatrix}$, $\begin{pmatrix} -1 & 1 & 1 \\ 1 & \mathbf{0} & \mathbf{0} \\ \mathbf{1} & \mathbf{0} & \mathbf{0} \end{pmatrix}$, $\begin{pmatrix} 0 & 0 & 1 \\ 1 & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{1} & \mathbf{0} \end{pmatrix}$, $\begin{pmatrix} 0 & 1 & 0 \\ 1 & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{1} \end{pmatrix}$ (numbers in bold are values of free variables, — notice that these correspond to standard unit vectors in $\mathbb{R}^5$.

3. (**a**) $M_{ef} = (e_1 \mid e_2)^{-1}(f_1 \mid f_2) = \begin{pmatrix} -87 & -7 \\ 25 & 2 \end{pmatrix}$. Coordinates of $\begin{pmatrix} 4 \\ 7 \end{pmatrix}$ relative to the first basis are $(e_1 \mid e_2)^{-1} \begin{pmatrix} 4 \\ 7 \end{pmatrix} = \begin{pmatrix} 16 \\ -3 \end{pmatrix}$, and coordinates of $\begin{pmatrix} 4 \\ 7 \end{pmatrix}$ relative to the second basis are $(f_1 \mid f_2)^{-1} \begin{pmatrix} 4 \\ 7 \end{pmatrix} = \begin{pmatrix} 11 \\ -139 \end{pmatrix}$.

(**b**) $M_{ef} = \begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix}^{-1} \begin{pmatrix} 3 & 1 & 0 \\ -1 & -1 & 2 \\ 1 & 0 & -1 \end{pmatrix} = \begin{pmatrix} -3/2 & -1 & 1/2 \\ 5/2 & 1 & -3/2 \\ 1/2 & 0 & 3/2 \end{pmatrix}$; coordinates are

$$\begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix}^{-1} \begin{pmatrix} 1 \\ 7 \\ -3 \end{pmatrix} = \begin{pmatrix} 3/2 \\ -9/2 \\ 11/2 \end{pmatrix}$$

and

$$\begin{pmatrix} 3 & 1 & 0 \\ -1 & -1 & 2 \\ 1 & 0 & -1 \end{pmatrix}^{-1} \begin{pmatrix} 1 \\ 7 \\ -3 \end{pmatrix} = \begin{pmatrix} 1/2 \\ -1/2 \\ 7/2 \end{pmatrix}$$

respectively.

4. (**a**) Indeed, addition of complex numbers and computing scalar multiples $z \mapsto a \cdot z$ clearly satisfies all axioms of vector spaces (because a real number a can be thought of as a complex number $a + 0i$, and we may use properties of complex numbers here). Also, $a + bi = a \cdot 1 + b \cdot i$, so this system of vectors is complete, and clearly linearly independent. Since $(3 - 7i) \cdot 1 = 3 - 7i$ and $(3 - 7i) \cdot i = 7 + 3i$, the matrix of this operator is $\begin{pmatrix} 3 & 7 \\ -7 & 3 \end{pmatrix}$.

(**b**) Indeed, addition of quaternions and computing scalar multiples $z \mapsto a \cdot z$ clearly satisfies all axioms of vector spaces (because a real number a can be thought of as a quaternion $a + 0i + 0j + 0k$, and we may use properties of quaternions here). Also, $a + bi + cj + dk = a \cdot 1 + b \cdot i + c \cdot j + d \cdot k$, so this system of vectors is complete, and clearly linearly independent. Since $(2+i-j+3k) \cdot 1 = 2+i-j+3k$, $(2+i-j+3k) \cdot i = -1+2i+3j+k$, $(2+i-j+3k) \cdot j = 1-3i+2j+k$, and $(2+i-j+3k) \cdot k = -3-i-j+2k$, the matrix of this operator is $\begin{pmatrix} 2 & -1 & 1 & -3 \\ 1 & 2 & -3 & -1 \\ -1 & 3 & 2 & -1 \\ 3 & 1 & 1 & 2 \end{pmatrix}$.

(**c**) Indeed, addition of quaternions and computing scalar multiples $z \mapsto a \cdot z$ clearly satisfies all axioms of vector spaces (because a real number a can be thought of as a matrix $a \cdot I$, and we may use properties of matrices here). Also, $\begin{pmatrix} a & b \\ c & d \end{pmatrix} = ae_1 + be_2 + ce_3 + de_4$, which shows that these matrices form a complete and linearly independent system. Computing the images of basis vectors, we have $e_1 \mapsto \begin{pmatrix} -1 & -1 \\ 1 & 0 \end{pmatrix}$, $e_2 \mapsto \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$, $e_3 \mapsto \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$, $e_4 \mapsto \begin{pmatrix} 0 & 1 \\ -1 & 1 \end{pmatrix}$, so the matrix of our operator is $\begin{pmatrix} -1 & -1 & 1 & 0 \\ -1 & 0 & 0 & 1 \\ 1 & 0 & 0 & -1 \\ 0 & 1 & -1 & 1 \end{pmatrix}$.

5. (**a**) We have $v \times (w_1 + w_2) = v \times w_1 + v \times w_2$ and $v \times (cw) = cv \times w$ by the known properties of cross products, so the operator is linear. Also, $v \times \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ -1 \\ -2 \end{pmatrix}$, $v \times \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}$, $v \times \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 2 \\ -1 \\ 0 \end{pmatrix}$, so the matrix of our operator is $\begin{pmatrix} 0 & 1 & 2 \\ -1 & 0 & -1 \\ -2 & 1 & 0 \end{pmatrix}$. It is skew-symmetric of odd order, so its determinant is equal to 0.

(**b**) We have $v \cdot (w_1 + w_2) = v \cdot w_1 + v \cdot w_2$ and $v \cdot (cw) = cv \times w$ by the known properties of dot products, so the operator is linear. Also, $v \cdot \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} = 1$, $v \cdot \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} = 2$, $v \cdot \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} = -1$, so the matrix of our operator is $\begin{pmatrix} 1 & 2 & -1 \end{pmatrix}$.