MA 1111/1212: Linear Algebra
Selected answers/solutions to the assignment due December 14, 2010

1. (**a**) $M_{ef} = (e_1 \mid e_2)^{-1}(f_1 \mid f_2) = \begin{pmatrix} -87 & -7 \\ 25 & 2 \end{pmatrix}$.

(**b**) $M_{ef}^{-1} \begin{pmatrix} 1 & -1 \\ 2 & 0 \end{pmatrix} M_{ef} = \begin{pmatrix} -1442 & -116 \\ 17938 & 1443 \end{pmatrix}$.

(**c**) We have $M_{ve} = (e_1 \mid e_2) = M_{ev}^{-1}$, so the matrix in question is

$$\begin{pmatrix} 1 & 4 \\ 1 & 3 \end{pmatrix} \begin{pmatrix} 1 & -1 \\ 2 & 0 \end{pmatrix} \begin{pmatrix} 1 & 4 \\ 1 & 3 \end{pmatrix}^{-1} = \begin{pmatrix} -28 & 37 \\ -22 & 29 \end{pmatrix}.$$

2. (**a**) Looking at the images of basis vectors, we immediately see that the matrix is $\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$.

(**b**) Using transition matrices, we immediately get $\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}^{-1} \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} -1 & -2 \\ 1 & 1 \end{pmatrix}$.

3. (**a**) $\begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}$ (**b**) $\begin{pmatrix} 1 & 0 & 0 \\ 2 & 0 & -1 \\ -1 & 1 & 2 \end{pmatrix}$

4. (**a**) Induction: for $n = 0$ it is true; if $\begin{pmatrix} 0 & 1 \\ -1 & 3 \end{pmatrix}^k \begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} b_k \\ b_{k+1} \end{pmatrix}$, then

$$\begin{pmatrix} 0 & 1 \\ -1 & 3 \end{pmatrix}^{k+1} \begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -1 & 3 \end{pmatrix} \begin{pmatrix} b_k \\ b_{k+1} \end{pmatrix} = \begin{pmatrix} b_{k+1} \\ 3b_{k+1} - b_k \end{pmatrix} = \begin{pmatrix} b_{k+1} \\ b_{k+2} \end{pmatrix}.$$

(**b**) Eigenvalues are roots of $t^2 - 3t + 1 = 0$, i.e. $\lambda_1 = \frac{3+\sqrt{5}}{2}$, $\lambda_2 = \frac{3-\sqrt{5}}{2}$. The corresponding eigenvectors are $\begin{pmatrix} 1 \\ \lambda_1 \end{pmatrix}$ and $\begin{pmatrix} 1 \\ \lambda_2 \end{pmatrix}$. Let $C = \begin{pmatrix} 1 & 1 \\ \lambda_1 & \lambda_2 \end{pmatrix}$. Then $C^{-1} \begin{pmatrix} 0 & 1 \\ -1 & 3 \end{pmatrix} C = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix}$, so $\begin{pmatrix} 0 & 1 \\ -1 & 3 \end{pmatrix}^n \begin{pmatrix} 0 \\ 1 \end{pmatrix} = C \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix}^n C^{-1} \begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} \frac{\lambda_1^n - \lambda_2^n}{\lambda_1 - \lambda_2} \\ \frac{\lambda_1^{n+1} - \lambda_2^{n+1}}{\lambda_1 - \lambda_2} \end{pmatrix}$, and $b_n = \frac{\lambda_1^n - \lambda_2^n}{\lambda_1 - \lambda_2}$.